



ELSEVIER

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



On general fibers of Gauss maps in positive characteristic



Katsuhisa Furukawa

Department of Mathematics, National Taiwan University, Taipei, Taiwan

ARTICLE INFO

Article history:

Received 16 July 2015

Available online 28 December 2015

Communicated by Steven Dale

Cutkosky

MSC:

primary 14N05

secondary 14M15

Keywords:

Gauss map

Degeneracy map

Strange variety

ABSTRACT

A general fiber of the Gauss map of a projective variety in $\mathbb{P}^N$ coincides with a linear subvariety of $\mathbb{P}^N$ in characteristic zero. In positive characteristic, S. Fukasawa showed that a general fiber of the Gauss map can be a non-linear variety. In this paper, we show that each irreducible component of such a possibly non-linear fiber of the Gauss map is contracted to one point by the degeneracy map, and is contained in a linear subvariety corresponding to the kernel of the differential of the Gauss map. We also show the inseparability of Gauss maps of strange varieties not being cones.

© 2015 Elsevier Inc. All rights reserved.

1. Introduction

Let $X \subset \mathbb{P}^N$ be an n -dimensional projective variety over an algebraically closed field of arbitrary characteristic, and let $\gamma : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ be the *Gauss map* of X , which sends a smooth point $x \in X$ to the embedded tangent space $T_x X$ to X at x in $\mathbb{P}^N$. We denote by $d_x \gamma : T_x X \rightarrow T_{\gamma(x)} \mathbb{G}(n, \mathbb{P}^N)$ the differential of γ at $x \in X$, a linear map between Zariski tangent spaces at x and $\gamma(x)$. We denote by $\text{rk } \gamma$ the rank of $d_x \gamma$ at general $x \in X$. Note that γ is separable if and only if $\text{rk } \gamma = \dim(\text{im}(\gamma))$.

E-mail address: katu@tims.ntu.edu.tw.

<http://dx.doi.org/10.1016/j.jalgebra.2015.11.042>

0021-8693/© 2015 Elsevier Inc. All rights reserved.

The *degeneracy map* κ of X is defined to be the rational map

$$\kappa : X \dashrightarrow \mathbb{G}(n - \operatorname{rk} \gamma, \mathbb{P}^N)$$

which sends a general point $x \in X$ to the $(n - \operatorname{rk} \gamma)$ -plane $\mathbb{L}_x(\ker d_x \gamma) \subset \mathbb{P}^N$, where we denote by $\mathbb{L}_x(A) \subset \mathbb{T}_x X$ the m -plane corresponding to an m -dimensional vector subspace $A \subset T_x X$.

In characteristic zero, it is well known that the closure of a general fiber of γ is equal to a linear subvariety of $\mathbb{P}^N$ (P. Griffiths and J. Harris [7, (2.10)], F.L. Zak [15, I, 2.3. Theorem (c)]; S.L. Kleiman and R. Piene gave another proof in terms of reflexivity [11, pp. 108–109]), where the fiber of γ is indeed equal to $\kappa(x) = \mathbb{L}_x(\ker d_x \gamma) \subset \mathbb{P}^N$ for general x in the fiber. The same statement holds in positive characteristic if γ is separable [4, Theorem 1.1].

A.H. Wallace [14, §7] pointed out that the Gauss map γ can be *inseparable* in positive characteristic. In this case, a general fiber of γ is not equal to $\kappa(x)$, since their dimensions are different. Moreover, it is possible that a general fiber of γ is *not* equal to a linear subvariety of $\mathbb{P}^N$; the fiber can be a union of points (Wallace [14, §7], Kleiman and A. Thorup [10, I-3], H. Kaji [8, Example 4.1], [9], J. Rathmann [13, Example 2.13], A. Noma [12]), and can be a union of non-linear varieties (S. Fukasawa [1, §7], [2], the author and A. Ito [5, §5], [6, Theorem 1.3]).

In this paper, we investigate the relationship between the $(n - \operatorname{rk} \gamma)$ -plane $\kappa(x) = \mathbb{L}_x(\ker d_x \gamma)$ and the general fiber of γ (possibly non-linear, as above).

Theorem 1.1. *Let $X \subset \mathbb{P}^N$ be an n -dimensional projective variety, and let $F \subset X$ be an irreducible component of the closure of a general fiber of the Gauss map γ . Then $\mathbb{L}_x(\ker d_x \gamma) \subset \mathbb{P}^N$ is constant on general $x \in F$ (in other words, F is contracted to one point by κ). Therefore F is contained in this constant $(n - \operatorname{rk} \gamma)$ -plane.*

The above constant $(n - \operatorname{rk} \gamma)$ -plane in $\mathbb{P}^N$ corresponds to $\kappa(F)$. We note that $\kappa(F_1) \neq \kappa(F_2)$ can occur for two irreducible components F_1 and F_2 of a general fiber of γ (see Example 3.4).

Next we examine Gauss maps of strange varieties. A projective variety $X \subset \mathbb{P}^N$ is said to be *strange* for a point $v \in \mathbb{P}^N$ if $v \in \mathbb{T}_x X$ holds for any smooth point $x \in X$. In previous studies of Gauss maps in positive characteristic, the inseparability of γ of strange X was often observed with attractive phenomena (e.g., [2,3]). Motivated by such observations, we show:

Theorem 1.2. *Let $X \subset \mathbb{P}^N$ be a projective variety which is strange for a point $v \in \mathbb{P}^N$. Then v is contained in $\mathbb{L}_x(\ker d_x \gamma)$. Moreover, if X is not a cone with vertex v , then γ is inseparable.*

The paper is organized as follows. In §2.1, we first show that the image $d_x \gamma(T_x X)$ in $T_{\gamma(x)} \mathbb{G}(n, \mathbb{P}^N)$ is constant for x in an irreducible component F of a general fiber of γ .

Download English Version:

<https://daneshyari.com/en/article/4583932>

Download Persian Version:

<https://daneshyari.com/article/4583932>

[Daneshyari.com](https://daneshyari.com)