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Restrictions of rainbow supercharacters

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A B S T R A C T

The maximal subgroup of unipotent upper-triangular matrices of the finite general linear groups is a fundamental family of p-groups. Their representation theory is well known to be wild, but there is a standard supercharacter theory, replacing irreducible representations by super-representations, that gives us some control over its representation theory. While this theory has a beautiful underlying combinatorics built on set partitions, the structure constants of restricted super-representations remain mysterious. This paper proposes a new approach to solving the restriction problem by constructing natural intermediate modules that help "factor" the computation of the structure constants. We illustrate the technique by solving the problem completely in the case of rainbow supercharacters (and some generalizations). Along the way we introduce a new q-analogue of the binomial coefficients that depend on an underlying poset.

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1. Introduction

Let N be a set with a total order so that we can construct the group $\mathrm{GL}_N(\mathbb{F}_q)$ of invertible matrices with rows and columns indexed by N and entries in the finite field $\mathbb{F}_q$ with q elements. The supercharacter theory of the finite unitriangular groups

$$\mathrm{UT}_N = \{u \in \mathrm{GL}_N(\mathbb{F}_q) \mid (u - \mathrm{Id}_N)_{ij} \neq 0 \text{ implies } i < j\},$$

where Id_N is the multiplicative identity of $\mathrm{GL}_N(\mathbb{F}_q)$, has developed into a rich combinatorics based on set partitions. In particular, [1] showed that — taken as a family — they give a representation theoretic realization of the Hopf algebra of symmetric functions in noncommuting variables (also studied in [17], for example), where the product comes from inflation and the coproduct from restriction. Thus, the representation theory of unipotent p-groups gives a noncommuting analogue to the classical combinatorial representation theory of the symmetric groups. The supercharacters of these groups give a new basis for this Hopf algebra.

One obstruction to making use of this connection is that while the inflation functor is straightforward for supercharacters, the restriction functor is still somewhat mysterious. That is, given a subset $K \subseteq N$, we want to decompose a supercharacter of UT_N as a linear combination of supercharacters of the subgroup

$$\mathrm{UT}_K = \{u \in \mathrm{UT}_N \mid (u - \mathrm{Id}_N)_{ij} \neq 0 \text{ implies } i, j \in K\} \subseteq \mathrm{UT}_N.$$

The paper [18] gives an iterative algorithm for computing restrictions of supercharacters, but this gives us little information about the coefficients that occur (though it does imply that they will be polynomial in the size of the underlying field q). As a preliminary step, [15] uses matchings in bipartite graphs to give a combinatorial characterization of when such a coefficient is nonzero; however, only a small set of examples have a direct computation of the coefficients.

The supercharacters of UT_N are indexed by set partitions of the set N. In this subject, it seems preferable to view set partitions as a set of pairs, as follows. Given a set partition $\mathrm{bl}(\lambda)$ of N, we can store the block information as a set of pairs

$$\lambda = \left\{ i \smile j \;\middle|\; \begin{array}{l} i < j \text{ with } i, j \text{ in the same block of } \mathrm{bl}(\lambda), \\ i < j' < j \text{ implies } j' \text{ is in a different block} \end{array} \right\}.$$

Let V^λ denote the UT_N-module whose trace is the supercharacter χ^λ indexed by λ (these modules are explicitly described in, for example, [12]). These modules have a convenient factorization

$$V^\lambda = \bigotimes_{i \smile j \in \lambda} V^{i \smile j}.$$

When restricting, these pairs then fall into three cases: