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Corrigendum

Corrigendum to “Dirac cohomology and translation functors” [J. Algebra 375 (2013) 328–336]

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ABSTRACT

The statements of Theorem 1.3, Lemma 3.2 and Proposition 5.2 in [4] are incorrect. We give counterexamples to these statements and we offer a replacement for Theorem 1.3, under stronger assumptions.

We first give a counterexample to Theorem 1.3 in [4]. We thank the referee for suggesting this counterexample.

Let G be the Hermitian group $SU(2, 1)$ with Cartan involution θ equal to the conjugate transpose inverse. The complexified Lie algebra of G is $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, the maximal compact subgroup corresponding to θ is $K = S(U(2) \times U(1))$ and the corresponding Cartan decomposition of $\mathfrak{g}$ is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We fix a compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be

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the space of diagonal matrices in $\mathfrak{g}$. The set Δ of $\mathfrak{h}$-roots in $\mathfrak{g}$ splits into subsets Δ_c of compact roots and Δ_n of non-compact roots.

Let $\mathfrak{b}$ be a θ-stable Borel subalgebra of $\mathfrak{g}$ not containing either of the two abelian K-invariant subspaces $\mathfrak{p}^{\pm}$ of $\mathfrak{p}$. Let $\Delta^+ \subset \Delta$ be the positive root system corresponding to $\mathfrak{b}$ and ρ the half sum of elements of Δ^+. The Harish-Chandra module of the non-holomorphic discrete series representation of G with Harish-Chandra parameter ρ is the cohomologically induced module $A_{\mathfrak{b}}(0)$. The Dirac cohomology $H_D(A_{\mathfrak{b}}(0))$ equals the irreducible finite-dimensional $\widetilde{K}$-module with highest weight ρ_n, where ρ_n is the half sum of elements of $\Delta^+ \cap \Delta_n$, see Proposition 5.4 in [2].

On the other hand, by Proposition 11.180 in [3], the limit of discrete series $A_{\mathfrak{b}}(-\rho)$ with infinitesimal character equal to 0 is a non-zero irreducible unitary $(\mathfrak{g}, K)$-module. The module $A_{\mathfrak{b}}(-\rho)$ can be obtained by translating $A_{\mathfrak{b}}(0)$ to infinitesimal character 0. It has trivial Dirac cohomology since no $W(\mathfrak{g}, \mathfrak{h})$-translate of the infinitesimal character 0 can be $\mathfrak{k}$-regular. Back to Theorem 1.3 in [4], we take $\lambda = 0$, $\nu = \rho$, $X_0 = A_{\mathfrak{b}}(-\rho)$, F_ρ the irreducible representation of G with highest weight ρ and $X_\rho = A_{\mathfrak{b}}(0)$. Since X_0 is a translate of X_ρ, it follows that X_ρ embeds into $X_0 \otimes F_\rho$ (see Proposition 7.143 in [3]). Since $H_D(X_\rho) \neq 0$ but $H_D(X_0) = 0$, we see that Theorem 1.3 of [4] does not hold.

Next we provide an example showing that statements of Lemma 3.2 and Proposition 5.2 are not correct. Let X_λ be a lowest weight discrete series module for $(\mathfrak{g}, K) = (\mathfrak{sl}(2, \mathbb{C}), SO(2))$. The K-types of X_λ are spanned by the weight vectors $x_{\lambda+1}, x_{\lambda+3}, \dots,$ where the subscripts denote the weights. Let F_ν be the finite-dimensional module with highest weight ν, spanned by the weight vectors $f_{-\nu}$, $f_{-\nu+2}, \dots, f_\nu$. Recall that the spin module is spanned by weight vectors $s_{\pm 1}$. Then one checks that

- $\operatorname{Ker} D_1^2 = \operatorname{Ker} D_1 = x_{\lambda+1} \otimes F_\nu \otimes s_{-1}$;
- $\operatorname{Ker} D_2 = X_\lambda \otimes f_{-\nu} \otimes s_{-1} \oplus X_\lambda \otimes f_\nu \otimes s_1$;
- $\operatorname{Ker} D_1 \cap \operatorname{Ker} D_2 = \mathbb{C}x_{\lambda+1} \otimes f_{-\nu} \otimes s_{-1}$.

($\operatorname{Ker} D_1^2 = \operatorname{Ker} D_1$ follows from unitarity of X_λ, or can be obtained by a direct calculation.) Assuming that $\lambda > \nu$, the translates of X_λ by F_ν are the lowest weight discrete series modules $X_{\lambda-\nu}$ and $X_{\lambda+\nu}$. One checks that $\varphi(\operatorname{Ker} D_{X_{\lambda-\nu}}) \subseteq \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2$, but

$$\varphi(\operatorname{Ker} D_{X_{\lambda+\nu}}) \nsubseteq \operatorname{Ker} D_1 \cap \operatorname{Ker} D_2.$$

This shows that Lemma 3.2 does not hold for $X_{\lambda+\nu}$. Moreover, $\varphi(\operatorname{Ker} D_{X_{\lambda+\nu}}) \nsubseteq \beta(\operatorname{Ker}(D_{X_\lambda}) \otimes \operatorname{Ker}(D_{F_\nu}) \otimes S^\star)$, so Proposition 5.2 also fails.

The mistake in the proof of Lemma 3.2 was the claim that $D_1 + D_2 = 0$ on $\varphi(\operatorname{Ker} D_{X_{\lambda+\nu}})$ implies $D_1^2 = D_2^2$ on $\varphi(\operatorname{Ker} D_{X_{\lambda+\nu}})$. Namely, $\varphi(\operatorname{Ker} D_{X_{\lambda+\nu}})$ need not be invariant under D_1 or D_2.

Note that in the above example Lemma 3.2 fails for one of the translates of X_λ, but it holds for the other translate. In the following, we show that a similar property holds in a much more general setting.