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Homaloidal determinants

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ABSTRACT

A form in a polynomial ring over a field is said to be homaloidal if its polar map is a Cremona map, i.e., if the rational map defined by the partial derivatives of the form has an inverse rational map. The overall object of this work is the search for homaloidal polynomials that are the determinants of sufficiently structured square matrices. We focus particularly on the generic square Hankel matrix and other specializations or degenerations of the fully generic square matrix. In addition to studying the homaloidal nature of these determinants, one establishes several results on the ideal theoretic invariants of the respective Jacobian ideals, such as primary components, multiplicity, reductions and free resolutions.

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Rees algebra
Plücker relations

1. Introduction

An important class of Cremona maps of $\mathbb{P}^n$ arises from the so-called *polar maps*, that is, rational maps whose coordinates are the partial derivatives of a homogeneous polynomial f in the homogeneous coordinate ring $R := k[x_0, \ldots, x_n]$ of $\mathbb{P}^n$. Geometrically, the relevance of such a map is that its indeterminacy locus is the singular locus of the corresponding hypersurface $V(f)$.

A homogeneous polynomial $f \in R$ whose polar map is a Cremona map is called *homaloidal* – though more often this designation applies to the corresponding hypersurface rather than to f itself. Unfortunately, there are scarcely any general methods to studying, much less recognizing, such polynomials. Results such as those in [12] and [22], although fascinating, are difficult to apply in practice and do not give a large picture. A more circumscribed environment consists of polynomials that are determinants of square matrices with homogeneous entries of the same degree. Alas, even for this class one lacks general methods, often happening that each such matrix requires a particular approach depending on its generic properties.

One of the goals of this paper is to consider structured matrices whose entries are variables of a polynomial ring over a field. Even for those there seems to be no comprehensive study of the homaloidal behavior of the corresponding determinants. Still, one advantage of dealing with these matrices is that they are often 1-generic in the sense of [10, Definition–Proposition 1.1]. This implies that their determinant is irreducible at the outset, thus allowing for a substantial class to search within.

We will assume throughout that the base field has characteristic zero. A good deal of intuition about the results gets lost in prime characteristic. Anyway, from the geometric point of view, the study of a polar map in characteristic zero is primevally driven by the properties of the Hessian determinant $h(f)$ of f, the reason being the classically known criterion for the dominance of the polar map in terms of the non-vanishing of the corresponding Hessian determinant. Although this criterion admits a vast generalization to arbitrary rational maps in terms of the Jacobian determinant of a basis of the corresponding linear system, it is in the polar case that the notion takes a fuller role.

Though the overall objective is to detect homaloidal determinants and their properties, we soon became aware of a richness of notions from commutative algebra that come alongside in a natural way. Often these notions and their use are crucial events for the geometric consequence. Thus, ideal theory in this paper is not just an aside, it is rather a live vein of the results. For example, a major underlying problem in this context is to understand the properties of the so-called *gradient ideal* (or *Jacobian ideal*) of the polynomial f, that is, the ideal $J = J(f) \subset R$ generated by the partial derivatives of f. The existence of tight lower bounds for the degrees of syzygies of the gradient ideal has been around in recent past, in connection with the so-called inverse Poincaré problem –