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Grothendieck categories of enriched functors

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It is shown that the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is Grothendieck whenever $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category and $\mathscr{C}$ is a category enriched over $\mathscr{V}$. Localizations in $[\mathscr{C}, \mathscr{V}]$ associated to collections of objects of $\mathscr{C}$ are studied. Also, the category of chain complexes of generalized modules $\mathrm{Ch}(\mathscr{C}_R)$ is shown to be identified with the Grothendieck category of enriched functors $[\mathrm{mod}\, R, \mathrm{Ch}(\mathrm{Mod}\, R)]$ over a commutative ring R, where the category of finitely presented R-modules $\mathrm{mod}\, R$ is enriched over the closed symmetric monoidal Grothendieck category $\mathrm{Ch}(\mathrm{Mod}\, R)$ as complexes concentrated in zeroth degree. As an application, it is proved that $\mathrm{Ch}(\mathscr{C}_R)$ is a closed symmetric monoidal Grothendieck model category with explicit formulas for tensor product and internal Hom-objects. Furthermore, the class of unital algebraic almost stable homotopy categories generalizing unital algebraic stable homotopy categories of Hovey–Palmieri–Strickland [14] is introduced. It is shown that the derived category of generalized modules $\mathscr{D}(\mathscr{C}_R)$ over commutative rings is a unital algebraic almost stable homotopy category which is not an algebraic stable homotopy category.

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1. Introduction

In the present paper we study categories of enriched functors

$$[\mathscr{C}, \mathscr{V}],$$

where $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category and $\mathscr{C}$ is a category enriched over $\mathscr{V}$ (i.e. a $\mathscr{V}$-category). The main result here states that the category $[\mathscr{C}, \mathscr{V}]$ is Grothendieck with an explicit collection of generators. Namely, the following theorem is true.

Theorem. *Let $\mathscr{V}$ be a closed symmetric monoidal Grothendieck category with a set of generators $\{g_i\}_I$. If $\mathscr{C}$ is a small $\mathscr{V}$-category, then the category of enriched functors $[\mathscr{C}, \mathscr{V}]$ is a Grothendieck $\mathscr{V}$-category with the set of generators $\{\mathscr{V}(c, -) \oslash g_i \mid c \in \mathrm{Ob}\,\mathscr{C}, i \in I\}$. Moreover, if $\mathscr{C}$ is a small symmetric monoidal $\mathscr{V}$-category, then $[\mathscr{C}, \mathscr{V}]$ is closed symmetric monoidal with explicit formulas for monoidal product and internal* Hom-*object.*

Taking into account this theorem, we refer to $[\mathscr{C}, \mathscr{V}]$ as a *Grothendieck category of enriched functors*. The usual Grothendieck category of additive functors

$$(\mathscr{B}, \mathrm{Ab})$$

from a pre-additive category $\mathscr{B}$ to abelian groups Ab is recovered from the preceding theorem in the case when $\mathscr{V} = \mathrm{Ab}$ ($\mathscr{B}$ is a $\mathscr{V}$-category). Further examples on how the category $[\mathscr{C}, \mathscr{V}]$ recovers some Grothendieck categories are given in Section 4.

The following result is an extension to enriched categories of similar results of [6,9].

Theorem. *Suppose $\mathscr{V}$ is a closed symmetric monoidal Grothendieck category. Let $\mathscr{C}$ be a $\mathscr{V}$-category and let $\mathscr{P}$ consist of a collection of objects of $\mathscr{C}$. Let $\mathscr{S}_{\mathscr{P}} = \{G \in [\mathscr{C}, \mathscr{V}] \mid G(p) = 0$ for all $p \in \mathscr{P}\}$. Then $\mathscr{S}_{\mathscr{P}}$ is a localizing subcategory of $[\mathscr{C}, \mathscr{V}]$ and $[\mathscr{P}, \mathscr{V}]$ is equivalent to the quotient category $[\mathscr{C}, \mathscr{V}]/\mathscr{S}_{\mathscr{P}}$.*

We apply Grothendieck categories of enriched functors to study homological algebra for generalized modules. The category of generalized modules

$$\mathscr{C}_R = (\mathrm{mod}\, R, \mathrm{Ab})$$

consists of the additive functors from the category of finitely presented R-modules, $\mathrm{mod}\, R$, to the category of abelian groups, Ab. Its morphisms are the natural transformations of functors. It is called the category of generalized R-modules for the reason that there is a fully faithful, right exact functor

$$M \mapsto - \otimes_R M$$

from the category of all R-modules to $\mathscr{C}_R$.