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Decomposing modular tensor products, and periodicity of 'Jordan partitions'

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ABSTRACT

Let J_r denote an $r \times r$ matrix with minimal and characteristic polynomials $(t-1)^r$. Suppose $r \leqslant s$. It is not hard to show that the Jordan canonical form of $J_r \otimes J_s$ is similar to $J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$ where $\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = rs$. The partition $\lambda(r,s,p) := (\lambda_1, \dots, \lambda_r)$ of rs, which depends only on r, s and the characteristic $p := \mathrm{char}(F)$, has many applications including the study of algebraic groups. We prove new periodicity and duality results for $\lambda(r,s,p)$ that depend on the smallest p-power exceeding r. This generalizes results of J.A. Green, B. Srinivasan, and others which depend on the smallest p-power exceeding the (potentially large) integer s. It also implies that for fixed r we can construct a finite table allowing the computation of $\lambda(r,s,p)$ for all s and p, with $s \geqslant r$ and p prime.

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1. Introduction

Consider a matrix whose minimal and characteristic polynomials equal $(t-1)^r$. To be explicit, take the $r \times r$ matrix J_r with 1s in positions (i,i) for $1 \leqslant i \leqslant r$, and $(i,i+1)$ for $1 \leqslant i < r$, and zeros elsewhere. Suppose $1 \leqslant r \leqslant s$. Then the Jordan canonical form of $J_r \otimes J_s$ is a direct sum $J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_r}$, with precisely r nonempty blocks, see Lemma 9(a). This decomposition depends on the characteristic p of the underlying field[3] F, and it determines a partition $\lambda(r,s,p) = (\lambda_1, \ldots, \lambda_r)$ of rs since $J_r \otimes J_s$ is an $rs \times rs$ matrix. We will assume that $\lambda_1 \geqslant \cdots \geqslant \lambda_r > 0$. The determination of this 'Jordan partition'[4] has applications to many significant problems. The representation theory of algebraic groups is governed by the behaviour of the unipotent elements, and indeed properties of $\lambda(r,s,p)$ are particularly useful (when $p > 0$) for the study of exceptional algebraic groups, see [14, 12]. More generally, Lindsey [13, Theorem 1] gives a useful (though somewhat technical) lower bound on the degree of the minimal faithful representation in characteristic p for certain groups with a prescribed Sylow p-subgroup structure. Lindsey's result, in turn, may be applied to the study of primitive permutation groups of p-power degree, see [19].

The most direct application, and the oldest, is to the study of modular representations of finite cyclic p-groups. Given two indecomposable modules V_r and V_s of a cyclic group G of order p^n, the module $V_r \otimes V_s$ is, by the Krull–Schmidt theorem, a sum of indecomposable modules $V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_r}$. Thus when $p > 0$, the partition $\lambda(r,s,p)$ arises naturally in this context too. The connection with matrices is straightforward: $G = \langle g \rangle$ has precisely p^n pairwise nonisomorphic indecomposable modules $V_1, \ldots, V_{p^n}$ which correspond to the matrix representations $G \to \mathrm{GL}(r, \mathbb{F}_p)\colon g \mapsto J_r$ where $1 \leqslant r \leqslant p^n$.

Definition 1. The following terminology will be used as convenient abbreviations.

(a) For integers r, s with $1 \leqslant r \leqslant s$, the ***standard partition*** $\lambda = (\lambda_1, \ldots, \lambda_r)$ of rs is the partition with $\lambda_i = r + s - 2i + 1$ for $1 \leqslant i \leqslant r$, i.e. $(s + r - 1, \ldots, s - r + 1)$.
(b) Call $\lambda = (\lambda_1, \ldots, \lambda_r)$ the **(r-)*uniform partition*** of rs if $\lambda_i = s$, for $1 \leqslant i \leqslant r$.
(c) The vector $\varepsilon(r,s,p) = (\varepsilon_1, \ldots, \varepsilon_r)$ with $\varepsilon_i = \lambda_i - s$, which measures the deviation of $\lambda(r,s,p) = (\lambda_1, \ldots, \lambda_r)$ from the uniform vector, is called the ***deviation vector***.
(d) The ***negative reverse*** of $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ is $\overline{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)} := (-\varepsilon_r, \ldots, -\varepsilon_2, -\varepsilon_1)$.
(e) The k-***multiple*** of $(\lambda_1, \ldots, \lambda_r)$ is the vector $(k\lambda_1, \ldots, k\lambda_1, \ldots, k\lambda_r, \ldots, k\lambda_r)$ of length kr where the size, and multiplicity, of each part is multiplied by k.

In characteristic zero, the partition $\lambda(r,s,0)$ was shown to be the standard partition independently by Aitken (1934), Roth (1934), and Littlewood (1936); for more background and references see [18, p. 416]. The change-of-basis matrix exhibiting the Jordan canonical form of $J_r \otimes J_s$ may be chosen to have rational entries, and so in 'large'

[3] We may assume that $F = \mathbb{F}_p$ or $\mathbb{Q}$ as the Jordan form of $J_r \otimes J_s$ is invariant under field extensions.
[4] This phrase was used by Dmitri Panyushev in the review MR2728146, but it is not used commonly.