Journal of Algebra 448 (2016) 104–126

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# On arboreal Galois representations of rational functions

Ashvin A. Swaminathan

*Department of Mathematics, Harvard College, 1 Oxford Street, Cambridge, MA 02138, USA*

A R T I C L E   I N F O

*Article history:*
Received 1 August 2014
Available online 3 December 2015
Communicated by Eva Bayer-Fluckiger

*MSC:*
11R32
37P15
11F80

*Keywords:*
Arithmetic dynamics
Dynamical sequences
Galois representations
Iterates
Rooted trees
Tree automorphisms

A B S T R A C T

The action of the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of a global field $K$ on a tree $T(\phi, \alpha)$ of iterated preimages of $\alpha \in \mathbb{P}^1(K)$ under $\phi \in K(x)$ with $\deg(\phi) \geq 2$ induces a homomorphism $\rho : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T(\phi, \alpha))$, which is called an *arboreal Galois representation*. In this paper, we address a number of questions posed by Jones and Manes [5,6] about the size of the group $G(\phi, \alpha) := \mathrm{im}\,\rho = \varprojlim_n \mathrm{Gal}(K(\phi^{-n}(\alpha))/K)$. Specifically, we consider two cases for the pair $(\phi, \alpha)$: (1) $\phi$ is such that the sequence $\{a_n\}$ defined by $a_0 = \alpha$ and $a_n = \phi(a_{n-1})$ is periodic, and (2) $\phi$ commutes with a nontrivial Möbius transformation that fixes $\alpha$.

In the first case, we resolve a question posed by Jones [5] about the size of $G(\phi, \alpha)$, and taking $K = \mathbb{Q}$, we describe the Galois groups of iterates of polynomials $\phi \in \mathbb{Z}[x]$ that have the form $\phi(x) = x^2 + kx$ or $\phi(x) = x^2 - (k+1)x + k$. When $K = \mathbb{Q}$ and $\phi \in \mathbb{Z}[x]$, arboreal Galois representations are a useful tool for studying the arithmetic dynamics of $\phi$. In the case of $\phi(x) = x^2 + kx$ for $k \in \mathbb{Z}$, we employ a result of Jones [4] regarding the size of the group $G(\psi, 0)$, where $\psi(x) = x^2 - kx + k$, to obtain a zero-density result for primes dividing terms of the sequence $\{a_n\}$ defined by $a_0 \in \mathbb{Z}$ and $a_n = \phi(a_{n-1})$.

In the second case, we resolve a conjecture of Jones [5] about the size of a certain subgroup $C(\phi, \alpha) \subset \mathrm{Aut}(T(\phi, \alpha))$ that contains $G(\phi, \alpha)$, and we present progress toward the proof

*E-mail address:* aaswaminathan@college.harvard.edu.

http://dx.doi.org/10.1016/j.jalgebra.2015.09.032

of a conjecture of Jones and Manes [6] concerning the size of $G(\phi, \alpha)$ as a subgroup of $C(\phi, \alpha)$.


## 1. Introduction

### 1.1. Background and definitions

Understanding which primes divide the Fermat numbers (i.e., numbers of the form $2^{2^n} + 1$) has been a question of interest for nearly four centuries. Progress toward answering related questions has been made with the tools of arithmetic dynamics, a field whose objective is to study the behavior of *dynamical sequences* $\{a_n\}$ that are defined by $a_n = \phi(a_{n-1})$ for some choice of a zeroth term $a_0 \in \mathbb{Z}$ and a polynomial $\phi \in \mathbb{Z}[x]$. In the case of the Fermat numbers, the relevance of arithmetic dynamics is evident, because the sequence $\{a_n\}$ of Fermat numbers can be expressed in recursive form as $a_0 = 3$ and $a_n = \phi(a_{n-1})$, where $\phi \in \mathbb{Z}[x]$ is given by $\phi(x) = (x-1)^2 + 1 = x^2 - 2x + 2$. Although it is difficult to give explicit descriptions of the primes that divide the terms of dynamical sequences, we may still be able to obtain qualitative results regarding the distribution of these prime factors in the set of all primes. In particular, given a choice of $a_0 \in \mathbb{Z}$ and $\phi \in \mathbb{Z}[x]$, we may attempt to determine the *natural density* $\mathrm{nd}(\phi, a_0)$ of the set

$$P_\phi(a_0) = \{\text{prime } p : p \mid a_n \text{ for at least one } n \geq 0\}$$

as a subset of the set of all primes, where $\mathrm{nd}(\phi, a_0)$ is defined by the limit

$$\mathrm{nd}(\phi, a_0) = \limsup_{x \to \infty} \frac{|\{p \in P_\phi(a_0) : p \leq x\}|}{|\{\text{prime } p \leq x\}|}.$$

In [7] and [8], Odoni showed that $\mathrm{nd}(\phi, a_0) = 0$ when the Galois groups $G_n(\phi)$ of iterates $\phi^n$ of $\phi$ satisfy the equality

$$\lim_{n \to \infty} \frac{|\{g \in G_n(\phi) : g \text{ fixes at least one root of } \phi^n\}|}{|G_n(\phi)|} = 0.$$

In the case of *Sylvester's sequence*, which is the sequence $\{a_n\}$ obtained by taking $a_0 = 2$ and $\phi(x) = x^2 - x + 1$, Odoni proved that $\mathrm{nd}(\phi, 2) = 0$ by showing that for each $n$ we have $G_n(\phi) \simeq \mathrm{Aut}(T_n)$, where $T_n$ is the complete, rooted binary tree of height $n$. To obtain this result for Sylvester's sequence, Odoni expressed $T_n$ as the tree $T_n(\phi, 0)$ of iterated preimages of 0 under $\phi$ (our notation for a preimage tree of height $n$ is $T_n(\text{“function”}, \text{“root”})$). More precisely, taking $\phi^0(x) = x$ and $\phi^{-n}(x) = \phi^{-1}(\phi^{-(n-1)}(x))$ for each $n \geq 1$, we can construct a binary tree $T(\phi, 0)$ of infinite height rooted at 0 such that the nodes on level $i$ are the elements of $\phi^{-i}(0)$ and such that for each $y \in \phi^{-i}(0)$,