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Counting integral points in certain homogeneous spaces

Dasheng Wei [a,b], Fei Xu [c,*]

[a] Academy of Mathematics and System Science, CAS, Beijing 100190, PR China
[b] Mathematisches Institute, Ludwig-Maximilians-Universität München, Theresienstr. 39, 80333 München, Germany
[c] School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China

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ABSTRACT

The leading term of asymptotic formula of the number of integral points in non-compact symmetric homogeneous spaces of semi-simple simply connected algebraic groups is given by the average of the product of the number of local solutions twisted by the Brauer–Manin obstruction. The similar result is also true for homogeneous spaces of reductive groups with some restriction. As application, we will give the explicit asymptotic formulae of the number of integral points of certain norm equations and prove the leading term of asymptotic formula of the number of integral matrices with a fixed irreducible characteristic polynomial over $\mathbb{Z}$ studied by Eskin–Mozes–Shah is equal to the product of the number of local integral solutions over all primes although the density function defined by Borovoi and Rudnick is not trivial in general. We also answer a question raised by Borovoi and Rudnick for comparing the number of integral symmetric matrices with the given determinant with the product of local densities.

* Corresponding author.
E-mail addresses: dshwei@amss.ac.cn (D. Wei), xufei@math.ac.cn (F. Xu).

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1. Introduction

The Hardy–Littlewood circle method is the classical method for counting integral points. Once this method can be applied, the leading term of asymptotic formula of the number of integral points will be given by the product of the number of local solutions and the local–global principle will be true (see [28]). However, the local–global principle does not hold in general. Borovoi and Rudnick in [4] counted the integral points in homogeneous spaces of semi-simple groups, where they measured the difference between the number of integral solutions and that of Hardy–Littlewood expectation by introducing so called the density functions. These density functions were described by using Kottwitz invariant and were not determined locally in [4]. By applying the recent achievement about strong approximation with Brauer–Manin obstruction developed in [7,12,8,36,37,3], we show that the asymptotic formula of the integral solutions will be given by the average of the product of local solutions twisted by the Brauer–Manin obstruction in this paper.

It should be pointed out that Manin's conjecture on asymptotic formulas for the number of rational points of bounded height on Fano varieties has been extensively studied for example in [24] and [1]. The integral version of such conjecture has been also studied in [5,6] and [32] (see also [20] and [21]). It is natural to ask if the leading term of such asymptotic formula can be precisely given by the average of product of local solutions twisted by the Brauer–Manin obstruction as well.

Notation and terminology are standard if not explained. Let F be a number field, o_F be the ring of integers of F, Ω_F be the set of all primes in F and ∞_F be the set of all infinite primes in F. We use $v < \infty_F$ to denote the finite primes of F. Let F_v be the completion of F at v and o_{F_v} be the local completion of o_F at v for each $v \in \Omega_F$. Write $o_{F_v} = F_v$ for $v \in \infty_F$. For any finite subset S of Ω_F containing ∞_F, the S-integers are defined to be elements in F which are integral outside S and denoted by o_S. We also denote the adeles of F by $\mathbb{A}_F$ and ideles of F by $\mathbb{I}_F = \mathbb{G}_m(\mathbb{A}_F)$ and $F_\infty = \prod_{v\in\infty_F} F_v$.

Let G be a connected reductive linear algebraic group and $\mathbf{X}$ be a separated scheme of finite type over o_F whose generic fiber

$$X = \mathbf{X} \times_{o_F} F \cong H\backslash G$$

is a right homogeneous space of G, where H is the stabilizer of the fixed point $P \in X(F)$. The map induced by the fixed point P is denoted by

$$p: \ G \longrightarrow X; \quad g \mapsto P \cdot g \tag{1.1}$$

The obvious necessary condition for $\mathbf{X}(o_F) \neq \emptyset$ is

$$\prod_{v\in\Omega_F} \mathbf{X}(o_{F_v}) \neq \emptyset \tag{1.2}$$