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Groups with countably many subgroups

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ABSTRACT

We describe soluble groups in which the set of all subgroups is countable and show that locally (soluble-by-finite) groups with this property are soluble-by-finite. Further, we construct a nilpotent group with uncountably many subgroups in which the set of all abelian subgroups is countable.

1. Introduction

It is a trivial observation that a group is finite precisely when it has only finitely many subgroups. As a next step we consider the question: When is the set of all subgroups of a group G countable? The abelian groups with this property have been characterised by

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Rychkov and Fomin in [13], see Lemma 2.3 below. We shall mostly be concerned with the analogous problem for soluble groups.

We shall say that a group G has the property CMS (or $G \in$ (CMS)) if the set $\mathcal{L}(G)$ of all subgroups of G is countable. Quite plainly all group with CMS are countable, and all groups with the maximal condition on subgroups also satisfy CMS.

The class (CMS) is closed under taking subgroups and quotients, but not under extensions, nor even direct products. For example, if p is a prime, the Prüfer p-group $\mathcal{C}_{p^\infty}$ has CMS, but its direct square $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$ has $2^{\aleph_0}$ subgroups. Groups with CMS satisfy a very strong rank-finiteness condition. Indeed, a direct product of infinitely many nontrivial groups certainly has at least $2^{\aleph_0}$ subgroups, therefore groups in (CMS) cannot have any such section. We shall prove that soluble groups without subnormal sections of the types excluded by these remarks satisfy CMS. Thus we shall see in Theorem 2.7 that *a soluble-by-finite group satisfies* CMS *if and only if it is minimax and has no (subnormal) sections of type* $\mathcal{C}_{p^\infty} \times \mathcal{C}_{p^\infty}$*, for any prime p*. We shall also see that *all locally (soluble-by-finite) groups with* CMS *are soluble-by-finite* (Theorem 2.12). A side feature of both theorems is that they also show that locally (soluble-by-finite) groups with uncountably many subgroups have at least $2^{\aleph_0}$ of them.

Let p be a prime. In the description of a soluble CMS-group G a role is played by the maximal number of factors isomorphic to $\mathcal{C}_{p^\infty}$ in a finite series of G. We shall denote this number by $\mathrm{r}_{p^\infty}(G)$. For instance the result by Rychkov and Fomin cited above could be rephrased by saying that the abelian groups with CMS are precisely those abelian minimax groups G such that $\mathrm{r}_{p^\infty}(G) \leq 1$ for all primes p. It turns out that $\mathrm{r}_{p^\infty}(G) \leq 2$ for all soluble groups G with CMS and all primes p; and, unlike the abelian case, $\mathrm{r}_{p^\infty}(G) = 2$ may occur, even if G is nilpotent.

On a different thread, we show—in Theorem 3.1—that *there exists a nilpotent group with uncountably many subgroups, of which just countably many are abelian.*

In some contrast with these results, nothing can be said on soluble groups in which the set of subgroups has a given uncountable cardinality. Indeed, it is not difficult to prove that if κ is an uncountable cardinal, then every abelian and hence every soluble group of cardinality κ has as many subnormal subgroups as subsets, i.e., 2^κ—it is proved in [3] that if the group is supposed to be nilpotent-by-finitely generated abelian then 'subnormal' may even be replaced by 'normal' in this statement. A consequence of these remarks and of Theorem 2.7 is that if G is an infinite soluble-by-finite group then $|\mathcal{L}(G)|$ is either $\aleph_0$ or $2^{|G|}$ (see Corollary 2.8); moreover, G has $|\mathcal{L}(G)|$ subnormal subgroups. Regarding the latter (hardly surprising) remark, note that there exist countable locally soluble groups with uncountably many subgroups but only countably many subnormal subgroups. Examples of this kind are the groups constructed in [17, (2.3)], in which all subnormal subgroups have finite index.

Leaving subnormality aside, Corollary 2.8 raises a question: What (non-trivial) restrictions are there (if any) on the possible cardinalities of the sets of all subgroups of infinite groups if the (generalised) solubility hypothesis is dropped? The following example is of interest in this regard. The construction in Theorem 35.2 of [9] provides a