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Model transition under local theta correspondence

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ABSTRACT

We study model transition for representations occurring in the local theta correspondence between split even special orthogonal groups and symplectic groups, over a non-archimedean local field of characteristic zero.

1. Introduction

Let F be a non-archimedean local field of characteristic zero. Let G be a reductive group defined over F. An important way to characterize irreducible admissible representations of $\mathrm{G}(F)$ is to consider various kinds of models. For example, non-degenerate Whittaker models of generic representations play significant roles in the theory of local factors of representations of $\mathrm{G}(F)$ and in the theory of automorphic forms. Given a reductive dual pair (G, H), and consider the local theta correspondence between

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representations of $\mathrm{G}(F)$ and of $\mathrm{H}(F)$, an interesting question is that how models of representations of $\mathrm{G}(F)$ and of $\mathrm{H}(F)$ are related under local theta correspondence.

In [5], Jiang, Nien and Qin proved that under the local theta correspondence, for certain representations, the generalized Shalika model on $\mathrm{SO}_{4n}(F)$ is corresponding to the symplectic linear model on $\mathrm{Sp}_{4n}(F)$, and conjectured that it is true for general representations (see [6, p. 542]). In this paper, we prove some results related to this conjecture (see Theorem 2.1). More precisely, we introduce generalized symplectic linear models on $\mathrm{Sp}_{4m}(F)$, which generalizes the symplectic linear models, and study the relations between the generalized Shalika models on $\mathrm{SO}_{4n}(F)$ and the generalized symplectic linear models on $\mathrm{Sp}_{4m}(F)$ under the local theta correspondence. A special case ($m = n$) of this result is proved by Hanzer [4] independently. We also introduce generalized Shalika models on $\mathrm{Sp}_{4n}(F)$ and generalized orthogonal linear models on $\mathrm{SO}_{4m}(F)$, and study the relations between them under the local theta correspondence.

Mœglin [8], Gomez and Zhu [3] studied the local theta lifting of generalized Whittaker models associated to nilpotent orbits. Note that the generalized Shalika models are indeed generalized Whittaker models associated to certain nilpotent orbits, but the generalized symplectic/orthogonal linear models are not. For example, the generalized Shalika models on $\mathrm{SO}_{4n}(F)$ are generalized Whittaker models associated to the nilpotent orbits parametrized by the partition $[2^{2n}]$. By [8] and [3], the full local theta lift on $\mathrm{Sp}_{2k}(F)$ has a nonzero generalized Whittaker model associated to a nilpotent orbit parametrized by the partition $[3^{2n}1^{\ell}]$, if $6n+\ell = 2k$. In general it is not known whether the small theta lift on $\mathrm{Sp}_{2k}(F)$ (if nonzero) would also carry this model.

This paper is organized as follows. In Section 2, we give the definitions for various models for representations of split even special orthogonal groups and symplectic groups, and introduce the main result Theorem 2.1. In Sections 3 and 4, we prove Part (1) and Part (2) of Theorem 2.1 respectively. In Section 5, we consider the converse of Theorem 2.1, and discuss some related results.

2. Models of representations

In this section, we define various models for representations of split even special orthogonal groups and symplectic groups.

For any positive integer k, let v_k be the $k \times k$ matrix with 1's in the second diagonal and zero's elsewhere. Let

$$\mathrm{SO}_{2\ell} = \{g \in \mathrm{GL}_{2\ell} \mid {}^t g v_{2\ell} g = v_{2\ell}\}$$

be the split even special orthogonal group. And let $SO_{2\ell} = \mathrm{SO}_{2\ell}(F)$. Let

$$\mathrm{Sp}_{2\ell} = \{g \in \mathrm{GL}_{2\ell} | {}^t g J_{2\ell} g = J_{2\ell}\}$$

be the symplectic group, where $J_{2\ell} = \begin{pmatrix} 0 & v_\ell \\ -v_\ell & 0 \end{pmatrix}$. And let $Sp_{2\ell} = \mathrm{Sp}_{2\ell}(F)$.