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Hadamard products of linear spaces

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A B S T R A C T

We describe properties of Hadamard products of algebraic varieties. We show any Hadamard power of a line is a linear space, and we construct star configurations from products of collinear points. Tropical geometry is used to find the degree of Hadamard products of other linear spaces.

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1. Introduction

The concept of Hadamard product, as matrix entry-wise multiplication, is well known in linear algebra: it has nice properties in matrix analysis [15,16,21,25] and has applications in both statistics and physics [16,22–24]. Recently, in the papers [8,9], the authors use this entry-wise multiplication to define a Hadamard product between projective varieties: given varieties $X, Y \subset \mathbb{P}^n$, their Hadamard product $X \star Y$ is the closure of the image of the rational map

$$X \times Y \dashrightarrow \mathbb{P}^n, \quad ([a_0 : \cdots : a_n], [b_0 : \cdots : b_n]) \mapsto [a_0 b_0 : a_1 b_1 : \ldots : a_n b_n].$$

For any projective variety X, we may consider its Hadamard square $X^{\star 2} = X \star X$ and its higher Hadamard powers $X^{\star r} = X \star X^{\star(r-1)}$. In [8], the authors use this definition to describe the algebraic variety associated to the restricted Boltzmann machine, which is the undirected graphical model for binary random variables specified by the bipartite graph $K_{r,n}$. This variety is the $r-$th Hadamard power of the first secant variety of $(\mathbb{P}^1)^n$. Note that [9] concerns the case $r = 2$, $n = 4$.

Hadamard products and powers are in fact well-connected to other operations of varieties. They are the multiplicative analogs of joins and secant varieties, and in tropical geometry, tropicalized Hadamard products equal Minkowski sums. It is natural to study properties of this new operation, and see its effect on various varieties. This paper is a first step in that direction. Here is how it is organized.

In Section 2, we start by giving a different definition of the Hadamard product of varieties in terms of projections of Segre products. As a first important result we give a Hadamard version of Terracini's Lemma (Lemma 2.12) which describes the tangent space of $X \star Y$ at $p \star q$ as the span $\langle p \star T_q(Y), q \star T_p(X)\rangle$. We also point out varieties parametrized by monomials are closed under Hadamard product.

In Section 3, our attention is fixed on the Hadamard powers of a line. This case is special and admits direct analysis by projective geometry. Theorem 3.4 is that the powers of a line are linear spaces, and our proof uses the Hadamard version of Terracini's Lemma. Proposition 3.6 then offers explicit equations.

In Section 4, we study the r-th square-free Hadamard power $Z^{\underline{\star} r}$ of a finite set Z of projective points. We obtain a classification when Z is collinear, using star configurations. In contrast with the standard approach to construct star configurations as intersections of a set of randomly chosen linear spaces, which can give points with complicated coordinates, Theorem 4.7 permits a cheaper construction, easily implementable in computer algebra software.

In Section 5, we recall basic definitions in tropical geometry, and give the precise connection to Hadamard products. Motivated by tropical considerations, we define a refined expected dimension formula for Hadamard products. Interestingly, Hadamard products can have deficient dimension (Example 5.5).