Journal of Algebra 448 (2016) 618–637

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The 290 fixed-point sublattices of the Leech lattice

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A R T I C L E I N F O

Article history:
Received 10 June 2015
Available online 10 November 2015
Communicated by Jon Carlson

Keywords:
Leech lattice
Fixed-point sublattice
Conway group

A B S T R A C T

We determine the orbits of fixed-point sublattices of the Leech lattice with respect to the action of the Conway group Co_0. There are 290 such orbits. Detailed information about these lattices, the corresponding coinvariant lattices, and the stabilizing subgroups, is tabulated in several tables.

1. Introduction

The *Leech lattice* Λ is the unique positive-definite, even, unimodular lattice of rank 24 without roots [30,13,2]. It may also be characterized as the most densely packed lattice in dimension 24 [11]. The group of isometries of Λ is the *Conway group* Co_0 [12]. For a subgroup $H \subseteq \mathrm{Co}_0$ we set

$$\Lambda^H = \{v \in \Lambda \mid hv = v \text{ for all } h \in H\}.$$

We call such a sublattice of Λ a *fixed-point sublattice*. Let $\mathcal{F}$ be the set of all fixed-point sublattices of Λ. The Conway group acts by translation on $\mathcal{F}$, because if $g \in \mathrm{Co}_0$,

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[1] Supported by NSF grant 1343609.

http://dx.doi.org/10.1016/j.jalgebra.2015.08.028
0021-8693/

then $g\Lambda^G = \Lambda^{gHg^{-1}}$. In this note, we classify the Co_0-orbits of fixed-point sublattices. We will prove:

Theorem 1.1. *Under the action of* Co_0, *there are exactly* 290 *orbits on the set of fixed-point sublattices of* Λ.

The purpose of the present note is not merely to enumerate the orbits of fixed-point sublattices, but to provide in addition a detailed analysis of their properties. In particular, this includes the *stabilizers* G, which are the (largest) subgroups of Co_0 that stabilize a given fixed-point sublattice pointwise. Information about the orbits of fixed-point lattices and their fixing groups is given in Table 1 in Section 4. Based on the theory that we present in Sections 2 and 3, this information was obtained by relying on extensive computer calculations using the computer algebra system MAGMA [6]. We shall say more about this in due course.

There are a number of reasons that make the classification of fixed-point lattices desirable. The group quotient $\mathrm{Co}_1 = \mathrm{Co}_0/\{\pm 1\}$ is one of the 26 sporadic simple groups. It contains 11 additional sporadic groups, 9 of which can be described in terms of lattice stabilizers. Although these particular realizations have been known for a long time, the complete picture that we provide is new.

The Leech lattice is also the starting point of the construction of interesting *vertex operator algebras* [3,22] and *generalized Kac–Moody Lie algebras*. Such Kac–Moody Lie algebras have root lattices that can often be described in terms of fixed-point lattices inside Λ [37], and the associated denominator identities provide Moonshine for the corresponding subgroups [4].

The *geometry of* $K3$ *surfaces* and certain *hyperkähler manifolds* X, over both the field of complex numbers and in finite characteristic, is controlled (using Torelli-type theorems) by lattices related to Λ. In this way, symmetry groups of X can be mapped into Co_0, and properties of the fixed-point lattices control which groups may appear. See [36,34,29,17] for $K3$ surfaces and [32,28,27] for other hyperkähler manifolds.

Much of the impetus for studying the finite symmetry groups of such manifolds, and recent developments in the related area of *Mathieu Moonshine* [21], came from the well-known theorem of Mukai [34]. This states that a finite group G of symplectic automorphisms of a K3 surface is isomorphic to a subgroup of the Mathieu group M_{23} with at least five orbits in its natural permutation representation on 24 letters; furthermore, there are just 11 subgroups (up to isomorphism) which are maximal subject to these conditions. A typical application of our results leads to a simplified approach to this theorem. Indeed, lattice-theoretic arguments [29,14] show that G can be embedded into Co_0 in such a way that $\mathrm{rk}\,\Lambda^G \geq 5$ and $\alpha(\Lambda^G) := \mathrm{rk}\,\Lambda^G - \mathrm{rk}\,A_{\Lambda^G} \geq 2$ (see Section 2 for notation). The containment $G \subseteq M_{23}$ follows immediately from Table 1, moreover the 11 maximal such groups are those G in Table 1 with $\mathrm{rk}\,\Lambda^G = 5$ and $\alpha(\Lambda^G) \geq 2$. The advantage of this approach compared to that of Kondō [29], who initiated the lattice-