Journal of Algebra 447 (2016) 31–55

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# 2-Engel relations between subgroups

M.P. Gállego [a], P. Hauck [b], M.D. Pérez-Ramos [c,*]

[a] *Departamento de Matemáticas, Universidad de Zaragoza, Edificio Matemáticas, Ciudad Universitaria, 50009 Zaragoza, Spain*

[b] *Fachbereich Informatik, Universität Tübingen, Sand 13, 72076 Tübingen, Germany*

[c] *Departament d'Àlgebra, Universitat de València, C/ Doctor Moliner 50, 46100 Burjassot (València), Spain*

A R T I C L E I N F O

*Article history:*
Received 30 July 2015
Available online 8 October 2015
Communicated by Gernot Stroth

*MSC:*
primary 20F45
secondary 20F18, 20D15, 20F12

*Keywords:*
2-Engel condition
2-generated subgroups
$\mathcal{L}$-connection
Nilpotency class

A B S T R A C T

In this paper we study groups $G$ generated by two subgroups $A$ and $B$ such that $\langle a, b\rangle$ is nilpotent of class at most 2 for all $a \in A$ and $b \in B$. A detailed description of the structure of such groups is obtained, generalizing the classical result of Hopkins and Levi on 2-Engel groups.



## 1. Introduction

A well-known result, usually attributed to Levi [19] though already proved by Hopkins [15] and in parts by Burnside [4], states that a group $G$ satisfying the 2-Engel condition $[x, y, y] = 1$ for all $x, y \in G$ is nilpotent of class at most 3 with $[G', G]$ of exponent

* Corresponding author.
*E-mail addresses:* pgallego@unizar.es (M.P. Gállego), hauck@informatik.uni-tuebingen.de (P. Hauck), Dolores.Perez@uv.es (M.D. Pérez-Ramos).

http://dx.doi.org/10.1016/j.jalgebra.2015.08.030
0021-8693/

dividing 3, where $G'$ denotes the derived subgroup of $G$. (For the history of this result and a presentation of the general theory of $n$-Engel groups we refer to the survey of Traustason [22].)

In the present paper we assume the 2-Engel condition only for certain pairs of elements. More precisely, we investigate groups generated by two subgroups $A$ and $B$ such that $[a, b, b] = [b, a, a] = 1$ for all $a \in A$ and $b \in B$. Of course, this is equivalent to saying that $\langle a, b\rangle$ is nilpotent of class at most 2 for all $a \in A$ and $b \in B$. In the terminology of [5] this property can be stated as $A$ and $B$ being $\mathcal{N}_2$-connected where $\mathcal{N}_2$ denotes the class of nilpotent groups of class at most 2. Therefore this paper can also be seen as a contribution to the study of $\mathcal{C}$-connected subgroups for various classes of groups $\mathcal{C}$ (cf. [2,6,8–10,14]).

For $\mathcal{N}_2$-connected subgroups $A$ and $B$, our main objective is to obtain information about the structure and embedding of $[A, B]$ and certain distinguished subgroups of $A$ and $B$.

One of the basic results is that $[A, B]$ centralizes $A'$ and $B'$ (Proposition 3.2 (i)). Moreover, $[A', B]$ centralizes $A$ whence, setting $G = \langle A, B\rangle$, $[A', B'] \leq Z(G)$ (Theorem 4.8 (i)).

Denoting by $A^2$ the group $\langle a^2 | a \in A\rangle$, we show in Theorem 4.1 and Theorem 4.8 that $[A^2, B] \leq Z([A, B]) \cap Z_3(G)$. In particular, if $A^2 = A$ or $B^2 = B$, the commutator $[A, B]$ is abelian. The same is true if $[A, B]'$ does not contain an element of order 2 (Corollary 4.3) or if $A$ and $B$ are normal in $G$ (Theorem 6.1). However, in general this need not be the case. In Section 7 we construct examples of finite 2-groups generated by $\mathcal{N}_2$-connected subgroups $A$ and $B$ where $[A, B]$ is nilpotent of arbitrarily large derived length (and hence also of arbitrarily large class).

It is easy to see that for infinite $\mathcal{N}_2$-connected subgroups $A$ and $B$ the commutator subgroup $[A, B]$ need not be nilpotent (Remark 4.7). In yet unpublished work, R. Dark has constructed examples of groups generated by two $\mathcal{N}_2$-connected elementary abelian subgroups $A$ and $B$, each of order 4, where in one of these $[A, B]$ is infinite (and hence not nilpotent) and in the other $[A, B]$ is finite and not nilpotent.

For finite groups, nilpotency of $[A, B]$ is equivalent to the subnormality of $A$ and $B$ in $G$ (see Theorem 2.3). In light of Dark's examples just mentioned it is interesting that for arbitrary $\mathcal{N}_2$-connected subgroups $A$ and $B$ of $G = \langle A, B\rangle$ certain characteristic subgroups of $A$ and $B$ are subnormal in $G$. It is shown in Theorem 5.1 that $A^2$ is subnormal in $G$ of defect at most 3, $A'$ is subnormal in $G$ of defect at most 2 and the higher terms of the lower central series of $A$ centralize $B$ and are therefore normal in $G$.

Theorem 5.3 is concerned with the intersection of $A$ and $B$: $A \cap B$ is a nilpotent subgroup of class at most 3; it is subnormal in $G$ of defect at most 3. Moreover, $[(A \cap B)', G]$ is a central elementary abelian 3-subgroup of $G$. For $A = B = G$, this is just the result of Hopkins and Levi.

We conclude these introductory remarks by pointing out that for normal $\mathcal{N}_2$-connected subgroups $A$ and $B$ significantly stronger statements are possible than those in the general situation. We mentioned already that in this case $[A, B]$ is abelian. Furthermore,