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The model theory of separably tame valued fields

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A B S T R A C T

A henselian valued field K is called separably tame if its separable-algebraic closure K^{sep} is a tame extension, that is, the ramification field of the normal extension $K^{\mathrm{sep}}|K$ is separable-algebraically closed. Every separable-algebraically maximal Kaplansky field is a separably tame field, but not conversely. In this paper, we prove Ax–Kochen–Ershov Principles for separably tame fields. This leads to model completeness and completeness results relative to the value group and residue field. As the maximal immediate extensions of separably tame fields are in general not unique, the proofs have to use much deeper valuation theoretical results than those for other classes of valued fields which have already been shown to satisfy Ax–Kochen–Ershov Principles. Our approach also yields alternate proofs of known results for separably closed valued fields.

1. Introduction

In this paper, we consider valued fields. By (K, v) we mean a field K equipped with a valuation v. We write a valuation in the classical additive (Krull) way, that is, the value

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group is an additively written ordered abelian group, the homomorphism property of v says that $vab = va+vb$, and the ultrametric triangle law says that $v(a+b) \geq \min\{va, vb\}$. Further, we have the rule $va = \infty \Leftrightarrow a = 0$. We denote the value group by vK, the residue field by Kv and the valuation ring by $\mathcal{O}_v$ or $\mathcal{O}_K$. For elements $a \in K$, the value is denoted by va, and the residue by av. By a valued field extension $(L|K, v)$ we mean that (L, v) is a valued field, $L|K$ is a field extension, and K is endowed with the restriction of v.

Our main concern is the model theory of separably tame valued fields, which we will introduce now.

A valued field is **henselian** if it satisfies Hensel's Lemma, or equivalently, if it admits a unique extension of the valuation to every algebraic extension field. The **henselization** of a valued field (L, v), denoted by $(L, v)^h$ or simply L^h, is the "minimal" extension of (L, v) which is henselian. It is unique up to isomorphism of valued fields. The henselization is an immediate separable-algebraic extension.

Every finite extension $(E|K, v)$ of valued fields satisfies the **fundamental inequality**:

$$n \geq \sum_{i=1}^{\mathrm{g}} \mathrm{e}_i \mathrm{f}_i \tag{1}$$

where $n = [E : K]$ is the degree of the extension, $v_1, \ldots, v_{\mathrm{g}}$ are the distinct extensions of v from K to E, $\mathrm{e}_i = (v_i E : vK)$ are the respective ramification indices and $\mathrm{f}_i = [Ev_i : Kv]$ are the respective inertia degrees. The extension is called **defectless** if equality holds in (1). A valued field (K, v) is called **defectless** (or **stable**) if each of its finite extensions is defectless, and **separably defectless** if each of its finite separable extensions is defectless. If char $Kv = 0$, then (K, v) is defectless (this is a consequence of the "Lemma of Ostrowski" [11, Section 2.2]). Note that $\mathrm{g} = 1$ if (K, v) is henselian, in which case (1) becomes $n \geq \mathrm{e}\,\mathrm{f}$. In particular, if a valued field (K, v) is henselian and defectless, then we have $n = \mathrm{e}\,\mathrm{f}$, i.e., $[E : K] = [vE : vK][Ev : Kv]$ for all finite extensions $(E|K, v)$.

Take a henselian field (K, v), and let p denote the **characteristic exponent** of its residue field Kv, i.e., $p = \mathrm{char}\ Kv$ if this is positive, and $p = 1$ otherwise. An algebraic extension $(L|K, v)$ of a henselian field (K, v) is called **tame** if every finite subextension $E|K$ of $L|K$ satisfies the following conditions:

(TE1) The ramification index $(vE : vK)$ is prime to p,
(TE2) The residue field extension $Ev|Kv$ is separable,
(TE3) The extension $(E|K, v)$ is defectless.

A **tame valued field** (in short, **tame field**) is a henselian field for which all algebraic extensions are tame. Equivalently, a tame field is one whose algebraic closure is equal to the ramification field K^r of the normal extension $K^{\mathrm{sep}}|K$, where K^{sep} denotes the separable-algebraic closure of K [11, Lemma 2.17(a)]. Likewise, a **separably tame field** is a henselian field for which all separable-algebraic extensions are tame. Equivalently, a separably tame field is one whose separable-algebraic closure is equal to K^r