Journal of Algebra 447 (2016) 183–205

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Multiple structures with arbitrarily large projective dimension supported on linear subspaces

Craig Huneke [a], Paolo Mantero [b], Jason McCullough [c,*], Alexandra Seceleanu [d]

[a] University of Virginia, Department of Mathematics, 141 Cabell Drive, Kerchof Hall, Charlottesville, VA 22904, USA
[b] University of Arkansas, Department of Mathematics, 301 SCEN, Fayetteville, AR 72701, USA
[c] Rider University, Department of Mathematics, 2083 Lawrenceville Road, Lawrenceville, NJ 08648, USA
[d] University of Nebraska–Lincoln, Department of Mathematics, 203 Avery Hall, Lincoln, NE 68588, USA

A R T I C L E   I N F O

Article history:
Received 23 January 2014
Available online 9 November 2015
Communicated by Steven Dale Cutkosky

MSC:
primary 13D05
secondary 14M06, 14M07, 13D02

Keywords:
Multiple structures
Projective dimension
Multiplicity
Primary ideals
Unmixed ideals
Free resolution
Linkage

A B S T R A C T

Let K be an algebraically closed field. There has been much interest in characterizing multiple structures in $\mathbb{P}^n_K$ defined on a linear subspace of small codimension under additional assumptions (e.g. Cohen–Macaulay). We show that no such finite characterization of multiple structures is possible if one only assumes unmixedness. Specifically, we prove that for any positive integers $h, e \geq 2$ with $(h, e) \neq (2, 2)$ and $p \geq 5$ there is a homogeneous ideal I in a polynomial ring over K such that (1) the height of I is h, (2) the Hilbert–Samuel multiplicity of R/I is e, (3) the projective dimension of R/I is at least p and (4) the ideal I is primary to a linear prime $(x_1, \ldots, x_h)$. This result is in stark contrast to Manolache's characterization of Cohen–Macaulay multiple structures in codimension 2 and multiplicity at most 4

* Corresponding author.
E-mail addresses: huneke@virginia.edu (C. Huneke), pmantero@uark.edu (P. Mantero), jmccullough@rider.edu (J. McCullough), aseceleanu2@math.unl.edu (A. Seceleanu).

http://dx.doi.org/10.1016/j.jalgebra.2015.09.019

and also to Engheta's characterization of unmixed ideals of height 2 and multiplicity 2.

1. Introduction

Let K be an algebraically closed field. We consider projective multiple (i.e. generically nonreduced) schemes whose reduced subschemes are linear subspaces in $\mathbb{P}^n_K$ for some n. Multiple structures in general have been widely studied with connections to vector bundles [12,1,14], Hartshorne's Conjecture [25], linkage theory [20] and set-theoretic complete intersections [24]. In our setting where the reduced subscheme is a smaller projective space, there are finite characterizations of multiple structures in codimension two in small degree and under certain hypotheses: Manolache gave structure theorems for scheme-theoretically Cohen–Macaulay multiple structures of degree at most 4 [15] and locally complete intersection multiple structures of degree at most 6 [16]. See [17] for a nice survey of these results.

The defining ideals of these schemes correspond to homogeneous ideals that are primary to a prime ideal generated by linear forms in a polynomial ring R over K. More broadly, we were interested in the homological structure of homogeneous unmixed ideals of any polynomial ring over K, that is, ideals whose associated primes all have the same height.

Engheta gave a complete characterization of unmixed ideals of height 2 and multiplicity 2:

Proposition 1.1. *(Engheta [10, Prop. 11].) Let R be a polynomial ring over an algebraically closed field and let $I \subset R$ be a height two unmixed ideal of multiplicity 2. Then $\mathrm{pd}(R/I) \leq 3$ and I is one of the following ideals.*

1. *A prime ideal generated by a linear form and an irreducible quadric.*
2. *$(x,y) \cap (x,z) = (x,yz)$ with independent linear forms x, y, z.*
3. *$(w,x) \cap (y,z) = (wy,wz,xy,xz)$ with independent linear forms w, x, y, z.*
4. *The (x,y)-primary ideal $(x,y)^2 + (ax+by)$ with independent linear forms x, y and forms $a, b \in \mathfrak{m}$ such that x, y, a, b form a regular sequence.*
5. *(x,y^2) with independent linear forms x, y.*

The hypothesis that K is algebraically closed is essential. Take for instance $R = \mathbb{Q}[w,x,y,z]$ and $P = (w^2+x^2, y^2+z^2, wz-xy, wy+xz)$. Then P is a prime ideal of height 2 and multiplicity 2, but is not degenerate (i.e. does not contain a linear form) as in case (1) above. Note that over $\mathbb{C}$, $P\mathbb{C}[w,x,y,z]$ is no longer prime but rather of type (3) since

$$P\mathbb{C}[w,x,y,z] = (w+ix, y+iz) \cap (w-ix, y-iz).$$