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Some remarks on the compressed zero-divisor graph

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ABSTRACT

Let R be a commutative ring with $1 \neq 0$. The zero-divisor graph $\Gamma(R)$ of R is the (undirected) graph with vertices the nonzero zero-divisors of R, and distinct vertices r and s are adjacent if and only if $rs = 0$. The relation on R given by $r \sim s$ if and only if $\mathrm{ann}_R(r) = \mathrm{ann}_R(s)$ is an equivalence relation. The compressed zero-divisor graph $\Gamma_E(R)$ of R is the (undirected) graph with vertices the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, and distinct vertices $[r]$ and $[s]$ are adjacent if and only if $rs = 0$. Let R_E be the set of equivalence classes for $\sim$ on R. Then R_E is a commutative monoid with multiplication $[r][s] = [rs]$. In this paper, we continue our study of the monoid R_E and the compressed zero-divisor graph $\Gamma_E(R)$. We consider several equivalence relations on R and their corresponding graph-theoretic translations to $\Gamma(R)$. We also show that the girth of $\Gamma_E(R)$ is three if it contains a cycle and determine the structure of $\Gamma_E(R)$ when it is acyclic and the monoids R_E when $\Gamma_E(R)$ is a star graph.

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1. Introduction

Let R be a commutative ring with $1 \neq 0$, and let $Z(R)$ be the set of zero-divisors of R. As in [8], the *zero-divisor graph* $\Gamma(R)$ of R is the (undirected) graph whose vertices are the elements of $Z(R) \setminus \{0\}$ such that distinct vertices r and s are adjacent if and only if $rs = 0$. The relationship between ring-theoretic properties of R and graph-theoretic properties of $\Gamma(R)$ has been studied extensively. For example, $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \leq 3$, $\mathrm{gr}(\Gamma(R)) \leq 4$ if $\Gamma(R)$ contains a cycle ([8, Theorems 2.3 and 2.4], [16, Theorem 1.6], [25, (1.4)]), and $\Gamma(R)$ is a finite graph with at least one vertex if and only if R is finite and not a field [8, Theorem 2.2].

Let S be a (multiplicative) commutative semigroup with 0, and let $Z(S)$ be the set of zero-divisors of S. As in [15], the *zero-divisor graph* $\Gamma(S)$ of S is the (undirected) graph whose vertices are the elements of $Z(S) \setminus \{0\}$ such that two distinct vertices x and y are adjacent if and only if $xy = 0$. Then $\Gamma(S)$ is connected with $\mathrm{diam}(\Gamma(S)) \leq 3$ [15, Theorem 1.2], and $\mathrm{gr}(\Gamma(S)) \leq 4$ if $\Gamma(S)$ contains a cycle (cf. [15, Theorem 1.5]).

For any elements r and s of R, define $r \sim s$ if and only if $\mathrm{ann}_R(r) = \mathrm{ann}_R(s)$. Then $\sim$ is an equivalence relation on R; for any $r \in R$, let $[r]_R = \{s \in R \mid r \sim s\}$. For example, it is clear that $[0]_R = \{0\}$, $[1]_R = R \setminus Z(R)$, and $[r]_R \subseteq Z(R) \setminus \{0\}$ for every $r \in R \setminus ([0]_R \cup [1]_R)$. Furthermore, the operation on the equivalence classes given by $[r]_R[s]_R = [rs]_R$ is well-defined (i.e., $\sim$ is a congruence relation on R) and thus makes the set $R_E = \{[r]_R \mid r \in R\}$ into a commutative monoid. Moreover, R_E is a commutative Boolean monoid if R is a reduced ring. The monoid R_E has been studied in [1,2,4–6].

As in [28], $\Gamma_E(R)$ will denote the (undirected) graph $\Gamma(R_E)$, called the *compressed zero-divisor graph of* R, whose vertices are the elements of $Z(R_E) \setminus \{[0]_R\} = R_E \setminus \{[0]_R, [1]_R\}$ such that distinct vertices $[r]_R$ and $[s]_R$ are adjacent if and only if $[r]_R[s]_R = [0]_R$, if and only if $rs = 0$. Thus, by our earlier remarks, $\Gamma_E(R)$ is connected with $\mathrm{diam}(\Gamma_E(R)) \leq 3$ and $\mathrm{gr}(\Gamma_E(R)) \leq 4$ if $\Gamma_E(R)$ contains a cycle. Note that if r and s are distinct adjacent vertices in $\Gamma(R)$, then $[r]_R$ and $[s]_R$ are adjacent in $\Gamma_E(R)$ if and only if $[r]_R \neq [s]_R$ (for example, this will always hold if R is reduced since if r and s are adjacent in $\Gamma(R)$ and $[r]_R = [s]_R$, then $r^2 = s^2 = 0$). In particular, $r - s - t - r$ is a triangle in $\Gamma(R)$ if and only if $[r]_R - [s]_R - [t]_R - [r]_R$ is a triangle in $\Gamma_E(R)$ when R is reduced.

In this paper, we continue our investigation of the monoid R_E and the compressed zero-divisor graph $\Gamma_E(R)$. In [4], we were interested in determining when $\Gamma_E(R) \cong \Gamma(S)$ for a commutative ring S with $1 \neq 0$. For example, given a commutative ring R with $1 \neq 0$ and $Z(R) \neq \{0\}$, it is shown that the mapping $\varphi_R : \Gamma(R) \to \Gamma_E(R)$ defined by $\varphi_R(r) = [r]_R$ is a bijection if and only if either R is a Boolean ring or $R \in \{\mathbb{Z}_4, \mathbb{Z}_2[X]/(X^2)\}$ [4, Theorem 2.9]. We also showed that for a reduced commutative ring R with $1 \neq 0$, there is a reduced commutative ring S with $1 \neq 0$ such that $\Gamma_E(R) \cong \Gamma(S)$ if and only if $T(R)$ is von Neumann regular [4, Theorem 4.3]. In [5], we examined when R_E can be embedded into R, either as a monoid or a partially ordered set, and we investigated when