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Uncountable groups with restrictions on subgroups of large cardinality ☆

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A B S T R A C T

The aim of this paper is to investigate the behaviour of uncountable groups of regular cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ belong to a given group class $\mathfrak{X}$. It is proved that if every proper subgroup of G of cardinality $\aleph$ has finite conjugacy classes, then also the conjugacy classes of G are finite, provided that G has no simple homomorphic images of cardinality $\aleph$. Moreover, it turns out that if G is a locally graded group of cardinality $\aleph$ in which every proper subgroup of cardinality $\aleph$ contains a nilpotent subgroup of finite index, then G is nilpotent-by-finite, again under the assumption that G has no simple homomorphic images of cardinality $\aleph$. A similar result holds also for uncountable locally graded groups whose large proper subgroups are abelian-by-finite.

1. Introduction

In a series of recent papers, it has been proved that if G is a (generalized) soluble group of infinite rank whose proper subgroups of infinite rank belong to a group class $\mathfrak{X}$, then

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G itself is an $\mathfrak{X}$-group, for several natural choices of the class $\mathfrak{X}$ (see for instance [8–10, 12]). Recall here that a group G is said to have *finite rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property; if such an r does not exist, we will say that the group G has *infinite rank*. These results suggest that the behaviour of *small* subgroups in a *large* group is neglectable, at least for a right choice of the definition of largeness and within a suitable universe.

The aim of this paper is to give a further contribution to this topic, by investigating uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ belong to a given group class $\mathfrak{X}$. The main obstacle here is a relevant result by S. Shelah [20], who proved (without appeal to the continuum hypothesis) that there exists a group of cardinality $\aleph_1$ whose proper subgroups (and even subsemigroups) have cardinality strictly smaller than $\aleph_1$. Observe that the situation is completely different in the case of fields, since it is well-known that any uncountable field contains a proper uncountable subfield (see for instance [7]). In order to avoid Shelah's example and other similar obstructions, we will use the additional requirement that the group has no simple homomorphic images of cardinality $\aleph$, a condition which is obviously satisfied in the case of locally soluble groups.

A group G is called an *FC-group* if every element of G has only finitely many conjugates, or equivalently if the centralizer $C_G(x)$ has finite index in G for each element x of G. The behaviour of groups with finite conjugacy classes has been investigated by several authors over the last seventy years, and we will use the monograph [21] as a general reference on the theory of FC-groups. The group G is said to be *minimal non-FC* if it is not an FC-group but all its proper subgroups have the FC-property. The structure of minimal non-FC groups has been completely described by V.V. Belyaev and N.F. Sesekin [2,3] in the case of groups admitting a non-trivial homomorphic image which is either finite or abelian. It is also easy to show that an arbitrary minimal non-FC group must be countable (see [21], Lemma 8.14). Moreover, it has been proved in [10] that if G is a soluble group of infinite rank whose proper subgroups of infinite rank have the FC-property, then G is an FC-group. Our first main result shows that if $\aleph$ is an uncountable regular cardinal and G is a group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the FC-property, then G itself is an FC-group, provided that it has no simple homomorphic images of cardinality $\aleph$. It follows that similar results hold for groups whose proper subgroups of large cardinality are either central-by-finite or finite-by-abelian.

The last section of the paper is devoted to the study of groups in which every proper subgroup of large cardinality contains a nilpotent subgroup of finite index. The structure of groups whose proper subgroups are nilpotent-by-finite was described by B. Bruno and R.E. Phillips, at least in the soluble case (see [5,6]). Furthermore, a classification of locally graded groups in which all proper subgroups are abelian-by-finite is known ([4], see also [1] and [18]). Recall here that a group G is called *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index; locally