Journal of Algebra 447 (2016) 424–431

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Kawashima's relations for interpolated multiple zeta values

Tatsushi Tanaka[a,*], Noriko Wakabayashi[b]

[a] Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Motoyama Kamigamo, Kita-ku, Kyoto-city, Kyoto 603-8555, Japan
[b] College of Science and Engineering, Ritsumeikan University, 1-1-1, Nojihigashi, Kusatsu-city, Shiga 525-8577, Japan

ARTICLE INFO

Article history:
Received 4 October 2014
Available online 9 November 2015
Communicated by Michel Van den Bergh

Keywords:
t-MZVs
Harmonic products
t-Kawashima relations
Cyclic sum formula

ABSTRACT

Recently, Yamamoto introduced polynomials in one variable t which interpolates multiple zeta and zeta-star values (t-MZVs for short), provided new prospects on two-one conjecture of Ohno and Zudilin and proved the cyclic sum formula for t-MZVs. In this paper, we establish a generalization of Kawashima's relations (t-Kawashima relations) for t-MZVs. We prove the cyclic sum formula for t-MZVs using a type of derivation operator, together with the t-Kawashima relations.

1. Introduction

Multiple zeta and zeta-star values are defined, for positive integers $k_1, \ldots, k_l$ with $k_1 \geq 2$, by convergent series

$$\zeta(k_1, \ldots, k_l) = \sum_{m_1 > \cdots > m_l > 0} \frac{1}{m_1^{k_1} \cdots m_l^{k_l}}, \quad \zeta^\star(k_1, \ldots, k_l) = \sum_{m_1 \geq \cdots \geq m_l \geq 1} \frac{1}{m_1^{k_1} \cdots m_l^{k_l}},$$

* Corresponding author.
E-mail addresses: t.tanaka@cc.kyoto-su.ac.jp (T. Tanaka), noriko-w@fc.ritsumei.ac.jp (N. Wakabayashi).

http://dx.doi.org/10.1016/j.jalgebra.2015.09.015
0021-8693/

abbreviated as MZVs and MZSVs, respectively. Decomposing the region of the sum of the defining series of MZSVs into 2^{n-1} regions by separating each '≥' into '>' and '=', we immediately find the (symbolic) invertible linear transformation between MZVs and MZSVs. For example,

$$\zeta^\star(k_1,k_2)=\zeta(k_1,k_2)+\zeta(k_1+k_2),$$

$$\zeta^\star(k_1,k_2,k_3)=\zeta(k_1,k_2,k_3)+\zeta(k_1+k_2,k_3)+\zeta(k_1,k_2+k_3)+\zeta(k_1+k_2+k_3).$$

Let t be a variable. In [11] Yamamoto introduced the following interpolation polynomial of MZVs and MZSVs.

$$\zeta^t(k_1,\ldots,k_l)=\sum_{\mathbf{p}} t^{l-\mathrm{dep}(\mathbf{p})}\zeta(\mathbf{p})\ (\in\mathbb{R}[t]), \tag{1}$$

where $\mathrm{dep}(\mathbf{p})$ is the depth (length) of $\mathbf{p}$ and $\mathbf{p}$ runs over all indices of the form

$$\mathbf{p}=(k_1\ \square\ \cdots\ \square\ k_l)$$

in which each $\square$ is filled by the comma, or the plus +. We call this polynomial t-MZV. Note that $\zeta^0=\zeta$ (MZV) and $\zeta^1=\zeta^\star$ (MZSV).

In this paper, we discuss certain quadratic formula which contains the cyclic sum formula for t-MZVs. In Section 2, we show the quadratic formula for t-MZVs as follows. Notations are described in next two sections.

Theorem 1. *For any $m\geq 1$ and any v, $w\in\mathfrak{H}_t y$, we have*

$$\sum_{\substack{p+q=m\\ p,q\geq 1}} Z_t(\varphi_t(v)\overset{t}{\circledast}(-tx+y)^{p-1}y)Z_t(\varphi_t(w)\overset{t}{\circledast}(-tx+y)^{q-1}y)$$
$$=-Z_t(\varphi_t(v\overset{t}{*}w)\overset{t}{\circledast}(-tx+y)^{m-1}y).$$

When $t=0$ and $t=1$, these formulas for MZVs and MZSVs are called Kawashima relations which first appeared in [5] (also see [8,9]). Kawashima relations are of importance since those for MZVs are expected to give whole relations among MZVs as well as extended double shuffle relations [4,7] or associator relations [1].

As an application, the following cyclic sum formula for t-MZVs is again proved. (The original appearance of the cyclic sum formula for MZVs and MZSVs are proved respectively in [3] and [6].)