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Lie-type-like groups ☆

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ABSTRACT

A finite simple group of Lie type in defining characteristic p has exactly two p-blocks, the principal block and a block of defect zero consisting of the Steinberg character whose degree is the p-part of the order of the group. In this paper we characterize finite groups G which have exactly the principal p-block and a p-block of defect zero consisting of an irreducible character of degree $|G|_p$.

1. Introduction

Determining the number of p-blocks of a finite group is in general a very subtle task and extremely difficult to answer if at all. The easiest case, namely that G has only

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one block, has been settled completely by Harris in [10]. The next case, i.e., a complete characterization of groups G which have exactly two blocks does not seem to be accessible. In particular, the situation $\mathrm{O}_p(G) \neq 1$ seems to be out of reach. However if we restrict to the case that G has only the principal p-block and a p-block of defect zero (which implies $\mathrm{O}_p(G) = 1$) the methods are strong enough to determine the group up to some extent. This case is of particular interest since the canonical situation is that of a finite simple group of Lie type in defining characteristic p (see [12], Section 8.5). In this case the block of defect zero consists of the Steinberg character which has degree $|G|_p$.

Definition 1.1. A finite group G is called Lie-type-like for the prime p if G has exactly the principal p-block and a p-block of defect zero with an irreducible ordinary character of degree $|G|_p$.

In this paper we prove the following two theorems where the second one depends on the classification of finite simple groups.

Theorem 1.2. *Let G be a finite group with $\mathrm{F}^*(G) = \mathrm{F}(G)$ and let p be a prime dividing $|G|$. Then G is a Lie-type-like group for the prime p if and only if $G = HP$ where H is an elementary abelian normal r-subgroup ($r \neq p$ a prime) and P is a Sylow p-subgroup of G which acts regularly on the non-trivial elements of H.*

All groups occurring in Theorem 1.2 can be classified easily which leads to the following consequence.

Corollary 1.3. *Let G be a finite group with $\mathrm{F}^*(G) = \mathrm{F}(G)$ and let p be a prime dividing $|G|$. Then G is a Lie-type-like group for the prime p if and only if one of the following holds.*

(i) *$G = HP \leq GL(m, 2)$, where $H = \mathrm{F}_2^m$ and P is a Singer cycle in $GL(m, 2)$ of order a Mersenne prime p acting regularly on the non-trivial elements of H.*
(ii) *$G = HP$, where $H = \mathrm{F}_r$ with r a Fermat prime and $P = \mathrm{F}_r^*$ acts on H by multiplication.*
(iii) *$G = S_3$ or $(C_3 \times C_3).Q_8$, where Q_8 is the quaternion group of order 8 acting regularly on the non-trivial elements of $C_3 \times C_3$.*

Proof. The "if" part is clear, and it suffices to prove the "only if" part. By Theorem 1.2, we write $G = HP$ with $|P| = p^n$ and $|H| = r^m$. Since P acts regularly on the non-trivial elements of H we have $r^m - 1 = p^n$.

If $m = 1 = n$, then $r = 3$, $p = 2$ and $G = S_3$.

If $n = 1 < m$, then p is a Mersenne prime and P (of order p) is generated by a Singer cycle in $GL(m, 2)$ acting on $H = \mathrm{F}_2^m$.

If $m = 1 < n$, then $r = |P| + 1 = 2^n + 1$ is a Fermat prime and $P = \mathrm{F}_r^*$ acts on $H = \mathrm{F}_r$ by multiplication.