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String C-groups as transitive subgroups of S_n

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ABSTRACT

If Γ is a string C-group which is isomorphic to a transitive subgroup of the symmetric group S_n (other than S_n and the alternating group A_n), then the rank of Γ is at most $n/2+1$, with finitely many exceptions (which are classified). It is conjectured that only the symmetric group has to be excluded.

1. Introduction

Classifications of string C-groups of high rank from almost simple groups has been a subject of interest for almost a decade now. Some striking results have been obtained, for instance for the symmetric groups. Leemans and Fernandes classified string C-groups of rank $n-1$ and $n-2$ for S_n [5,8] and more recently, they extended this classification

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to rank $n-3$ and $n-4$ with Mixer [9]. The key of such a classification was first to show that, from a certain value of n onwards, all the maximal parabolic subgroups of a string C-group of rank $r \geq n-4$ must be intransitive. This was done by bounding the rank of string C-groups constructed from transitive groups of degree n that are not S_n nor A_n. These ideas motivated the present paper where we deal with transitive groups in a more general way than in the papers cited above. Another motivation is to contribute to the proof of the following conjecture.

Conjecture 1.1. *(See [7, Conjecture 9.1].) Let $n \geq 12$. The highest rank of an abstract regular polytope having A_n as automorphism group is $\lfloor (n-1)/2 \rfloor$.*

Suppose that $\rho_0, \ldots, \rho_{d-1}$ are involutions which generate a string C-group Γ. This means that

- if $|i-j| > 1$, then ρ_i and ρ_j commute (the *string property*);
- if Γ_I denotes the group generated by $\{\rho_i : i \in I\}$, for $I \subseteq \{0, \ldots, d-1\}$, then

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$$

(the *intersection property*).

It is known that string C-groups are the same thing as automorphism groups of regular polytopes [13, Section 2E]. The number d is the *rank* of the string C-group (or of the polytope).

Suppose that Γ is a string C-group of rank d which is a subgroup of the symmetric group S_n.

- If $\Gamma = \mathrm{S}_n$, then $d \leq n-1$, with equality if and only if the generators ρ_i are the *Coxeter generators* of S_n (that is, $\rho_i = (i+1, i+2)$ for $0 \leq i \leq d-1$) when $n \neq 4$, or equivalently, the polytope is the regular simplex. When $n = 4$, the hemicube gives another set of generators. Moreover, the polytopes (or string C-groups) of ranks $n-2$, $n-3$ and $n-4$ with $\Gamma = \mathrm{S}_n$ have been classified [5,9].
- Examples of string C-groups of rank $\lfloor (n-1)/2 \rfloor$ with $\Gamma = \mathrm{A}_n$ have been found [6,7]. It is conjectured that this is the largest possible rank for $n \geq 12$ (see above).
- Ranks arbitrarily close to n can be realised by string C-groups corresponding to intransitive subgroups of S_n. For example, if $n = n_1 + \cdots + n_k$, where $n_i > 1$ for all i, then take Coxeter generators for $\mathrm{S}_{n_1}, \ldots, \mathrm{S}_{n_k}$: we obtain a string C-group $\Gamma = \mathrm{S}_{n_1} \times \cdots \times \mathrm{S}_{n_k}$ of rank $n-k$ by ordering the factors arbitrarily.

The groups not covered by these remarks are the transitive subgroups of S_n other than S_n and A_n. We prove the following theorem:

Theorem 1.2. *Let Γ be a string C-group of rank d which is isomorphic to a transitive subgroup of S_n other than S_n or A_n. Then one of the following holds:*