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On varieties of groups satisfying an Engel type identity

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Let m, n be positive integers, v a multilinear commutator word and $w = v^m$. Denote by $v(G)$ and $w(G)$ the verbal subgroups of a group G corresponding to v and w, respectively. We prove that the class of all groups G in which the w-values are n-Engel and $w(G)$ is locally nilpotent is a variety (Theorem A). Further, we show that in the case where m is a prime-power the class of all groups G in which the w-values are n-Engel and $v(G)$ has an ascending normal series whose quotients are either locally soluble or locally finite is a variety (Theorem B). We examine the question whether the latter result remains valid with m allowed to be an arbitrary positive integer. In this direction, we show that if m, n are positive integers, u a multilinear commutator word and v the product of 896 u-words, then the class of all groups G in which the v^m-values are n-Engel and the verbal subgroup $u(G)$ has an ascending normal series whose quotients are either locally soluble or locally finite is a variety (Theorem C).

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1. Introduction

A variety is a class of groups defined by equations. More precisely, if W is a set of words, the class of all groups satisfying the laws $W \equiv 1$ is called the variety determined by W. By a well-known theorem of Birkhoff [9, 2.3.5], varieties are precisely classes of groups closed with respect to taking subgroups, quotients and Cartesian products of their members. Some interesting varieties of groups have been discovered in the context of the Restricted Burnside Problem solved in the affirmative by Zelmanov [18,19].

It is well-known that the solution of the Restricted Burnside Problem is equivalent to each of the following statements.

(i) The class of locally finite groups of exponent n is a variety.
(ii) The class of locally nilpotent groups of exponent n is a variety.

The equivalence of the two above results follows from the famous Hall–Higman reduction theorem [5]. Recall that a group is said to locally have some property if all its finitely generated subgroups have that property. A number of varieties of (locally nilpotent)-by-soluble groups were presented in [10,11].

The solution of the Restricted Burnside Problem strongly impacted our understanding of Engel groups. An element x of a group G is called a *(left) Engel element* if for any $g \in G$ there exists $n = n(x, g) \geq 1$ such that $[g, {}_n x] = 1$, where the commutator $[g, {}_n x]$ is defined recursively by the rule

$$[g, {}_n x] = [[g, {}_{n-1} x], x]$$

starting with $[g, {}_0 x] = g$ and $[g, {}_1 x] = [g, x] = g^{-1}x^{-1}gx$. If n can be chosen independently of g, then x is a (left) n-Engel element. A group G is called n-Engel if all elements of G are n-Engel. In [17] Zelmanov remarked that the eventual solution of the Restricted Burnside Problem would imply that the class of locally nilpotent n-Engel groups is a variety (see also Wilson [16]). Recently groups with n-Engel word-values were considered [1,2,14,15].

If w is a word in variables $x_1, \dots, x_k$ we think of it primarily as a function of k variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by the values of w. The word w is called *multilinear commutator* (of weight s) if it has a form of a multilinear Lie monomial (in precisely s independent variables). Particular examples of multilinear commutators are the lower central words γ_k, defined by

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \dots, x_k], \quad \text{for } k \geq 2;$$

and the derived words δ_k, on 2^k variables, which are defined by

$$\delta_0 = x_1, \quad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})], \quad \text{for } k \geq 1.$$

The verbal subgroups $\gamma_k(G)$, corresponding to γ_k, and $G^{(k)}$, corresponding to δ_k, are the k-th term of the lower central series of G and the k-th derived subgroup of G, respectively.