Journal of Algebra 447 (2016) 516–529

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The structure of preenvelopes with respect to maximal Cohen–Macaulay modules

Hiroki Matsui

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan

A R T I C L E   I N F O

Article history:
Received 30 April 2015
Available online 9 November 2015
Communicated by Kazuhiko Kurano

MSC:
13C14
13C60

Keywords:
Envelope
Special preenvelope
Maximal Cohen–Macaulay module

A B S T R A C T

This paper studies the structure of special preenvelopes and envelopes with respect to maximal Cohen–Macaulay modules. We investigate the structure of them in terms of their kernels and cokernels. Moreover, using this result, we also study the structure of special proper coresolutions with respect to maximal Cohen–Macaulay modules over a Henselian Cohen–Macaulay local ring.

1. Introduction

Throughout this paper, we assume that $(R, \mathfrak{m}, k)$ is a d-dimensional Cohen–Macaulay local ring with canonical module ω. All R-modules are assumed to be finitely generated. Denote by $\mathsf{mod}R$ the category of finitely generated R-modules and by MCM the full subcategory of $\mathsf{mod}R$ consisting of maximal Cohen–Macaulay R-modules.

We define $(-)^\dagger := \mathsf{Hom}_R(-, \omega)$ and $\delta_M : M \to M^{\dagger\dagger}$ as a natural homomorphism for an R-module M. Note that if M is maximal Cohen–Macaulay, δ_M is an isomorphism.

E-mail address: m14037f@math.nagoya-u.ac.jp.

http://dx.doi.org/10.1016/j.jalgebra.2015.09.042

Let $\mathcal{X}$ be a full subcategory of $\mathsf{mod}R$. The notion of $\mathcal{X}$-(pre)covers and $\mathcal{X}$-(pre)envelopes have been playing an important role in the representation theory of algebras; see [4–6,8,9,13] for instance. For $\mathcal{X} = \mathsf{MCM}$, a celebrated theorem due to Auslander and Buchweitz [3] says that for any R-module M, there exists a short exact sequence

$$0 \to Y \xrightarrow{f} X \xrightarrow{\pi} M \to 0$$

where X is maximal Cohen–Macaulay and Y has finite injective dimension. The map π is called a maximal Cohen–Macaulay approximation of M. Then π is an MCM-precover of M, and is an MCM-cover if Y and X have no non-zero common direct summand via f. If R is Henselian, every R-module has an MCM-cover; see [12,15].

In this paper we mainly study the MCM-envelope, and the MCM-preenvelope which is called special. A result of Holm [10, Theorem A] states that every R-module has a special MCM-preenvelope, and if R is Henselian, every R-module has an MCM-envelope. It is natural to ask when a given homomorphism is a special MCM-preenvelope or an MCM-envelope, and we give an answer to this question by using the kernels and cokernels. Our first main result is the following theorem.

Theorem 1.1. *Let $\mu : M \to X$ be an R-homomorphism such that X is maximal Cohen–Macaulay.*

(1) *The following are equivalent.*
   - (a) *μ is a special MCM-preenvelope of M.*
   - (b) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$ *and* $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.
   - (c) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, *and there exists an exact sequence* $0 \to S \to \mathsf{Coker}\,\mu \to T \to U \to 0$ *such that*
     - $\mathsf{codim}\,S > 1$,
     - $\mathsf{codim}\,U > 2$,
     - *T satisfies (S_2),*
     - *$T^\dagger$ has finite injective dimension and satisfies (S_3).*

(2) *The following are equivalent if R is Henselian.*
   - (a) *μ is an MCM-envelope of M.*
   - (b) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$ *and* $\mathsf{Coker}(\mu^{\dagger\dagger})$ *has no free summand.*
   - (c) $\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, *and there exists an exact sequence* $0 \to S \to \mathsf{Coker}\,\mu \xrightarrow{u} T \to U \to 0$ *such that*
     - $\mathsf{codim}\,S > 1$,
     - $\mathsf{codim}\,U > 2$,
     - *T satisfies (S_2),*
     - *$T^\dagger$ has finite injective dimension and satisfies (S_3),*
     - $\mathsf{Im}\,u$ *has no non-zero free summand.*