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# Lattices from Hermitian function fields ☆

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ABSTRACT

We consider the well-known Rosenbloom–Tsfasman function field lattices in the special case of Hermitian function fields. We show that in this case the resulting lattices are generated by their minimal vectors, provide an estimate on the total number of minimal vectors, and derive properties of the automorphism groups of these lattices. Our study continues previous investigations of lattices coming from elliptic curves and finite Abelian groups. The lattices we are faced with here are more subtle than those considered previously, and the proofs of the main results require the replacement of the existing linear algebra approaches by deep results of Gerhard Hiss on the factorization of functions with particular divisor support into lines and their inverses.



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## 1. Introduction

A lattice is a discrete subgroup in a Euclidean space $\mathbb{R}^n$. Lattice theory aims to understand geometric properties of lattices and to use them for a variety of applications, such as discrete optimization problems or coding theory. Some of the geometrically most interesting lattices, in particular those possessing many symmetries, come from several well-studied algebraic constructions. These include, for instance, ideal lattice constructions from number fields and polynomial rings; see, e.g., [1,2] and [8], respectively, for a detailed overview of these constructions. A series of prominent algebraic constructions of lattices are also presented in [14]. In this paper, we focus our attention on an important algebraic construction, originally introduced by Rosenbloom and Tsfasman in [10] and later described in [14], that of *function field lattices.*

The construction of function field lattices given in [14] is as follows. Let $F$ be an algebraic function field (of a single variable) with the finite field $\mathbb{F}_q$ as its full field of constants, where $q$ is a prime power. Let $\mathcal{P} = \{P_0, P_1, P_2, \ldots, P_{n-1}\}$ be the set of rational places of $F$. For each place $P_i$, let $v_i$ denote the corresponding normalized discrete valuation and let $\mathcal{O}_{\mathcal{P}}^*$ be the set of all nonzero functions $f \in F$ whose divisor has support contained in the set $\mathcal{P}$. Then $\mathcal{O}_{\mathcal{P}}^*$ is an Abelian group, $\sum_{i=0}^{n-1} v_i(f) = 0$ for each $f \in \mathcal{O}_{\mathcal{P}}^*$, and we define

$$\deg f := \sum_{v_i(f)>0} v_i(f) = \frac{1}{2}\sum_{i=0}^{n-1} |v_i(f)|.$$

Let $\varphi_{\mathcal{P}} : \mathcal{O}_{\mathcal{P}}^* \to \mathbb{Z}^n$ be the group homomorphism given by

$$\varphi_{\mathcal{P}}(f) = (v_0(f), v_1(f), \ldots, v_{n-1}(f)).$$

Then $L_{\mathcal{P}} := \mathrm{Image}(\varphi_{\mathcal{P}})$ is a finite-index sublattice of the root lattice

$$A_{n-1} = \left\{ \mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{Z}^n : \sum_{i=0}^{n-1} x_i = 0 \right\}$$

with minimum distance

$$d(L_{\mathcal{P}}) \geq \min\left\{ \sqrt{2\deg f} : f \in \mathcal{O}_{\mathcal{P}}^* \setminus \mathbb{F}_q \right\}, \tag{1}$$

and

$$\det L_{\mathcal{P}} \leq \sqrt{n} h_F \leq \sqrt{n}\left(1 + q + \frac{n-q-1}{g}\right)^g, \tag{2}$$

where $g$ is the genus of $F$ and $h_F$ is the divisor class number of $F$, that is, the size of the group of divisor classes of $F$ of degree 0, denoted by $\mathrm{Cl}^0(F)$. Here, as in [14], we