Journal of Algebra 447 (2016) 664–676

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Koszul–Young flattenings and symmetric border rank of the determinant

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ARTICLE INFO

Article history:
Received 19 May 2015
Available online 18 November 2015
Communicated by Reinhard Laubenbacher

Keywords:
Computational complexity
Determinant
Permanent

ABSTRACT

We present new lower bounds for the symmetric border rank of the $n \times n$ determinant for all n. Further lower bounds are given for the 3×3 permanent.

1. Introduction

The determinant polynomial is ubiquitous, its properties have been extensively studied. However basic questions regarding its complexity are still not understood. Lower bounds for the (symmetric) border rank of a polynomial provide a measurement of its complexity and, as such, have become an area of growing interest. In this paper we use techniques developed in [12] to explore this question. We prove a new lower bound for the symmetric border rank of the $n \times n$ determinant.

Definition 1.1. Let V be a vector space and let $S^d V$ denote homogeneous degree d polynomials on V^*. Given $P \in S^d V$, define its symmetric rank $R_s(P)$ by

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http://dx.doi.org/10.1016/j.jalgebra.2015.11.011
0021-8693/

$$R_s(P) = \min\left\{ r \in \mathbb{N} : P = \sum_{i=1}^{r} (v_i)^d, v_i \in V \right\}.$$

Symmetric rank is not semi-continuous under taking limits or Zariski closure, so we introduce symmetric border rank.

Definition 1.2. Let $P \in S^dV$. Define the symmetric border rank of P, $\underline{R}_s(P)$ to be

$$\underline{R}_s(P) = \min\left\{ r \in \mathbb{N} : P \in \overline{\{T : R_s(T) = r\}} \right\}$$

where the overline denotes Zariski closure.

Theorem 1.3. *For $n \geq 5$, the following are lower bounds on the symmetric border rank of the determinant, $\underline{R}_s(\det_n)$.*

For n even:

$$\underline{R}_s(\det_n) \geq \left(1 + \tfrac{8(-8+6n^2+n^3)}{(-1+n)(2+n)(4+n)^2(-2+n^2)}\right) \binom{n}{\frac{n}{2}}^2.$$

For n odd:

$$\underline{R}_s(\det_n) \geq \left(1 + \tfrac{16(9+8n+n^2)}{(3+n)(5+n)^2(-2+n^2)}\right) \binom{n}{\frac{n-1}{2}}^2.$$

Remark 1.4. Previously known lower bounds were

$$\underline{R}_s(\det_n) \geq \tbinom{n}{\frac{n}{2}}^2$$

for n even, and

$$\underline{R}_s(\det_n) \geq \tbinom{n}{\frac{n-1}{2}}^2$$

for n odd.

Remark 1.5. Asymptotically, our bound is

$$\underline{R}_s(\det_n) \gtrsim \tfrac{2^{2n+1}}{\pi \cdot n} + \tfrac{2^{2n+1}}{\pi \cdot n^4}$$

whereas the previous lower bounds are approximately $\underline{R}_s(\det_n) \gtrsim \frac{2^{2n+1}}{\pi \cdot n}$.

Theorem 1.6. $\underline{R}_s(\det_4) \geq 38$*.*

Remark 1.7. The previous bound was $\underline{R}_s(\det_4) \geq 36$.

Using a Macaulay2 [8] package developed by Steven Sam [14], we also show