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On zeros of a system of quadratic forms in 3 variables

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ABSTRACT

Let k be a field of characteristic distinct from 2, f_1, f_2, f_3 quadratic forms in 3 variables over k. We prove that these forms have a common zero over k if and only if the quadratic form $u_1 f_1 + u_2 f_2 + u_3 f_3$ is isotropic over the field $k(u_1, u_2, u_3)$ and the cubic form $\det(t_1 f_1 + t_2 f_2 + t_3 f_3)$ in variables t_1, t_2, t_3 is isotropic over k. We also consider a similar question for n quadratic forms in 3 variables where $n \geq 4$.

Let k be a field of characteristic different from 2. By a quadratic form over k we mean a homogeneous polynomial of degree 2 in several variables. In the sequel we will omit the word "quadratic" when speaking of quadratic forms, and call them just forms. However, besides quadratic forms we will also deal with *cubic* forms.

As usual, we call an n-dimensional form f over k isotropic if there exists a k-rational point $(a_1 : \cdots : a_n) \in \mathbb{P}_k^{n-1}$ such that $f(a_1, \ldots, a_n) = 0$. Otherwise the form f is called anisotropic. We will say that the forms $f_1, \ldots, f_m$ in n variables have a common zero

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over k if there exists a k-rational point $(a_1 : \cdots : a_n) \in \mathbb{P}_k^{n-1}$ such that $f_i(a_1, \dots, a_n) = 0$ for each $1 \le i \le m$. Obviously, a necessary condition for the existence of a common zero for the forms $f_1, \dots, f_m$ is isotropy of the form $u_1 f_1 + \cdots + u_m f_m$ over the field $k(u_1, \dots, u_m)$, where $u_1, \dots, u_m$ are indeterminates. Brumer's theorem [1] states that two forms f_1, f_2 have a common zero over k if and only if the form $u_1 f_1 + u_2 f_2$ over $k(u_1, u_2)$ is isotropic (or, equivalently, the form $f_1 + t f_2$ is isotropic over $k(t)$). This means that the condition above is also sufficient in the case of two quadratic forms. On the other hand, Pfister noted that the strict analog of Brumer's theorem is not valid for *three* forms even in *three* variables ([7] and [8], Ch. 9, example 1.9). In other words, given 3-dimensional forms f_1, f_2, f_3, the isotropy of the form $u_1 f_1 + u_2 f_2 + u_3 f_3$ over the field $k(u_1, u_2, u_3)$ does not imply in general the existence of a common zero of the forms f_1, f_2, f_3. In fact, there are plenty of similar examples of negative nature. For instance, in [2] one can find an example of 7 forms f_i in 10 variables over $k = \mathbb{Q}$ without a common zero, but such that the form $u_1 f_1 + \cdots + u_7 f_7$ is isotropic. In [3] there are examples of 3 forms in 4 variables with the similar property.

The key point in all these examples is the existence of an odd-degree field extension l/k such that the forms f_i have a common zero over l. Then Springer's theorem [5,9] implies that the form $u_1 f_1 + \cdots + u_m f_m$ is isotropic.

However, in this note we prove the following analog of Brumer's theorem in the case of 3 forms in 3 variables:

Theorem 1. *Let k be a field, f_1, f_2, f_3 quadratic forms over k in three variables. Then the following two conditions are equivalent:*

1) *The forms f_1, f_2, f_3 have a common zero over k.*
2) *The form $u_1 f_1 + u_2 f_2 + u_3 f_3$ is isotropic over $k(u_1, u_2, u_3)$, and the cubic form $\det(t_1 f_1 + t_2 f_2 + t_3 f_3)$ in the variables t_1, t_2, t_3 is isotropic over k.*

Proof. 1) $\Longrightarrow$ 2) Since the forms f_1, f_2, f_3 have a common zero over k, it is obvious that the form $u_1 f_1 + u_2 f_2 + u_3 f_3$ is isotropic over $k(u_1, u_2, u_3)$. Furthermore, by a linear change of variables we may assume that

$$f_1(0,0,1) = f_2(0,0,1) = f_3(0,0,1) = 0.$$

Then

$$f_i(x, y, z) = g_i(x, y) + a_i xz + b_i yz$$

for some $a_i, b_i \in k$, where g_i are forms in variables x, y $(1 \le i \le 3)$. Choose $\alpha_1, \alpha_2, \alpha_3 \in k$ not simultaneously zero such that

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = 0.$$