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# Alternating quiver Hecke algebras

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For simply-laced, reversible quivers, we consider the fixed-point subalgebra of the quiver Hecke algebra under the homogeneous sign map. This leads to a new family of algebras we call alternating quiver Hecke algebras. We give a basis theorem and a presentation by generators and relations which is strikingly similar to the KLR presentation for quiver Hecke algebras.



## Introduction

The study of quiver Hecke algebras is a recent development in representation theory, dually motivated from studying representations of cyclotomic Hecke algebras and Khovanov–Lauda–Rouquier (KLR) algebras. Whereas cyclotomic Hecke algebras, being related to symmetric groups and classical Iwahori–Hecke algebras, have been studied for some time, KLR algebras are relative newcomers, and the two ostensibly different families were linked by groundbreaking work of Brundan and Kleshchev [4]. Because of

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this deep and meaningful link, it is becoming customary to refer to KLR algebras as quiver Hecke algebras. In recent years much work has been done to study these algebras in their own right. Mathas [12] gives a particularly erudite survey of such developments.

Quiver Hecke algebras are associative graded algebras whose presentation by generators and relations, and whose grading, depends on the data of a quiver. Particularly interesting is the case when the quiver is simply laced, as quotients of these algebras are isomorphic to cyclotomic Hecke algebras, endowing these algebras with a $\mathbb{Z}$-grading. In this paper, we define a new family of algebras called alternating quiver Hecke algebras. For quiver Hecke algebras whose quivers are simply laced and reversible (isomorphic as quivers to their opposite), it is possible to define a homogeneous involution on the quiver Hecke algebra; studying the fixed-point subalgebra under this involution gives rise to our family of algebras. We discuss these algebras using a version of Clifford theory for associative algebras, and construct a homogeneous basis and a presentation by homogeneous generators and relations which are reminiscent of Khovanov and Lauda [10] and Rouquier's [15] theorems for quiver Hecke algebras. Cyclotomic quotients of these algebras are studied in [2] and [3].

This paper is organised as follows. We start by giving the definition of quiver Hecke algebras for arbitrary quivers. Then we discuss the Clifford theory for associative algebras that will give the technical mechanism for most of our proofs; this relies on the construction of the opposite quiver. In section 3 we discuss alternating quiver Hecke algebras, giving a basis theorem for this new family of algebras. Finally, in section 4 we prove the main result of this paper, giving a KLR-style presentation for alternating quiver Hecke algebras.

## 1. Quiver Hecke algebras and opposite quivers

In this section we define quiver Hecke algebras, and give some basic properties. These algebras were introduced by Khovanov and Lauda [10] and Rouquier [15].

Let $\Gamma$ be a simply-laced quiver with vertex set $I$. Following Kac [9], to the quiver $\Gamma$ we attach the usual Lie theoretic data of the positive roots $\{\alpha_i \mid i \in I\}$, the fundamental weights $\{\Lambda_i \mid i \in I\}$, the non-degenerate pairing $(\Lambda_i, \alpha_j) = \delta_{ij}$, for $i, j \in I$, and the Cartan matrix $C = (c_{ij})_{i,j \in I}$ where

$$c_{ij} = \begin{cases} 2, & \text{if } i = j, \\ -1, & \text{if } i \leftarrow j \text{ or } i \rightarrow j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

Let $P^+ = \bigoplus_{i \in I} \mathbb{N}\Lambda_i$ and $Q^+_\Gamma = \bigoplus_{i \in I} \mathbb{N}\alpha_i$. The height of $\beta = \sum_i a_i\alpha_i \in Q^+_\Gamma$ is the non-negative integer $\operatorname{ht}\beta = \sum_i a_i \in \mathbb{N}$. Fix $n \geq 0$ and let $Q^+_n = \{\beta \in Q^+_\Gamma \mid \operatorname{ht}\beta = n\}$. For $\beta \in Q^+_n$ let

$$I^\beta = \{\mathbf{i} = (i_1, \ldots, i_n) \in I^n \mid \beta = \alpha_{i_1} + \ldots + \alpha_{i_n}\}. \tag{1.2}$$