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# Twisted strong Macdonald theorems and adjoint orbits

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A B S T R A C T

The strong Macdonald theorems state that, for $L$ reductive and $s$ an odd variable, the cohomology algebras $H^*(L[z]/z^N)$ and $H^*(L[z,s])$ are freely generated, and describe the cohomological, $s$-, and $z$-degrees of the generators. The resulting identity for the $z$-weighted Euler characteristic is equivalent to Macdonald's constant term identity for a finite root system. We calculate $H^*(\mathfrak{p}/z^N\mathfrak{p})$ and $H^*(\mathfrak{p}[s])$ for $\mathfrak{p}$ a standard parahoric in a twisted loop algebra, giving strong Macdonald theorems that take into account both a parabolic component and a possible diagram automorphism twist. In particular we show that $H^*(\mathfrak{p}/z^N\mathfrak{p})$ contains a parabolic subalgebra of the coinvariant algebra of the fixed-point subgroup of the Weyl group of $L$, and thus is no longer free. We also prove a strong Macdonald theorem for $H^*(\mathfrak{b}; S^*\mathfrak{n}^*)$ and $H^*(\mathfrak{b}/z^N\mathfrak{n})$ when $\mathfrak{b}$ and $\mathfrak{n}$ are Iwahori and nilpotent subalgebras respectively of a twisted loop algebra. For each strong Macdonald theorem proved, taking $z$-weighted Euler characteristics gives an identity equivalent to Macdonald's constant term identity for the corresponding affine root system. As part of the proof, we study the regular adjoint orbits for the adjoint action of the twisted arc group associated to $L$, proving an analogue of the Kostant slice theorem.



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**Contents**



## 1. Introduction

Macdonald's constant term identity states that if $\Delta$ is a reduced root system then

$$[e^0] \prod_{\alpha\in\Delta^+} \prod_{i=1}^{N} (1 - q^{i-1}e^{-\alpha})(1 - q^i e^{\alpha}) = \prod_{i=1}^{l} \binom{N(m_i+1)}{N}_q, \tag{1}$$

where $m_1, \dots, m_l$ is the list of exponents of $L$ and $\binom{a}{b}_q$ is the $q$-binomial coefficient. Macdonald presented the identity as a conjecture in [32], and observed that it constitutes the untwisted case of a constant term identity for affine root systems. Further extensions (including a $(q,t)$-version) and proofs for individual affine root systems followed (see for instance [4,19,40,38,41,34,18,17,26]) until Cherednik gave a uniform proof of the most general version using double affine Hecke algebras [7].

Suppose $\Delta$ is the root system of a semisimple Lie algebra $L$ with exponents $m_1, \dots, m_l$. Prior to Cherednik's proof, Hanlon observed in [20] that the constant term identity would follow from a stronger conjecture: