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Conway–Coxeter friezes and beyond: Polynomially weighted walks around dissected polygons and generalized frieze patterns

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A B S T R A C T

Conway and Coxeter introduced frieze patterns in 1973 and classified them via triangulated polygons. The determinant of the matrix associated to a frieze table was computed explicitly by Broline, Crowe and Isaacs in 1974, a result generalized 2012 by Baur and Marsh in the context of cluster algebras of type A. Higher angulations of polygons and associated generalized frieze patterns were studied in a joint paper with Holm and Jørgensen. Here we take these results further; we allow arbitrary dissections and introduce polynomially weighted walks around such dissected polygons. The corresponding generalized frieze table satisfies a complementary symmetry condition; its determinant is a multisymmetric multivariate polynomial that is given explicitly. But even more, the frieze matrix may be transformed over a ring of Laurent polynomials to a nice diagonal form generalizing the Smith normal form result given in [3]. Considering the generalized polynomial frieze in this context it is also shown that the non-zero local determinants are monomials that are given explicitly, depending on the geometry of the dissected polygon.

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1. Introduction

Conway and Coxeter [6,7] defined arithmetical friezes by a local determinant condition and classified them via triangulated polygons; the friezes have a glide reflection symmetry. The geometry of these triangulations was then studied by Broline, Crowe and Isaacs [4]; they associated a matrix to these which was shown to be symmetric, and they computed its determinant to be $-(-2)^{n-2}$, for any triangulated n-gon.

These notions have found renewed interest in recent years in the context of cluster algebras, see for example [1,2,5]. Indeed, Baur and Marsh [2] generalized the determinant formula in the context of cluster algebras of type A, to a version involving cluster variables.

Another generalization and refinement of the results on Conway–Coxeter friezes was investigated recently in joint work with Holm and Jørgensen on d-angulations of polygons [3]. Again, a matrix associated to such d-angulations with entries counting suitable paths turned out to be symmetric, and its determinant and Smith normal forms were computed; these are independent of the particular d-angulation. We have also generalized the notion of friezes in this context by weakening the local determinant condition. It should also be pointed out that in recent work by Holm and Jørgensen, generalized friezes were categorified via a modified Caldero–Chapoton map [8,9].

Here the investigations of [3] are taken further in two combinatorial directions. On the one hand, instead of d-angulations we study arbitrary polygon dissections; the results from [3] have natural generalizations in this wider context. Furthermore, going far beyond the arithmetical results, we do not only count suitable paths along the polygons, but we consider weighted walks around the dissected polygons.

The corresponding associated weight matrices then have (multivariate) polynomial entries; they are not quite symmetric, but satisfy a natural complementarity condition (see Theorem 3.5 and Corollary 4.3). Surprisingly, as seen in the arithmetical situation, the determinant only depends on the pieces of the dissection but not on the specific way the pieces are glued together in the dissection; an explicit formula for the determinant of a weight matrix is given in Theorem 4.4. In fact, we are even able to provide an equivalent diagonal form of the weight matrix over a suitable ring of Laurent polynomials.

As in the arithmetical situation, we associate a generalized frieze pattern to a dissected polygon; this is the frieze of weight polynomials, which only has a translation symmetry due to the complementarity condition. Furthermore, the local 2×2 determinants 0 and 1 in the generalized arithmetical friezes are seen to be specializations of 0's and explicit monomials in the generalized polynomial friezes. The condition when we have a nonzero determinant and the explicit description of the monomials are given in Theorem 5.1.

2. From arcs around triangulated polygons to walks around dissected polygons

A frieze in the sense of Conway and Coxeter [6,7] is a pattern of natural numbers arranged in bi-infinite interlaced rows as in the example further below where the top