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On generalized $\mathfrak{U}$-hypercentral subgroups of a finite group ☆

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We give some conditions under which a normal subgroup of a finite group G is contained in the intersection of all maximal supersoluble subgroups of G.

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1. Introduction

In this paper, all groups are finite and G always denotes a finite group. Moreover, p is always supposed to be a prime and π denotes a set of primes. A subgroup H of G is said to be *S-quasinormal* or *S-permutable* in G if $HP = PH$ for all Sylow subgroups P of G. A subgroup H of G is said to be a *quaternionic subgroup* (L.A. Shemetkov) of G if G has a section A/B such that $H \leq A$, $H \cap B = 1$ and $A/B \cong Q_8$.

Recall that a *subgroup functor* is a function τ which assigns to each group G a set of subgroups $\tau(G)$ of G satisfying that $1 \in \tau(G)$ and $\theta(\tau(G)) = \tau(\theta(G))$ for any isomorphism $\theta : G \to G^*$. If $H \in \tau(G)$, then we say that H is a *τ-subgroup* of G. Throughout this paper, τ always denotes a subgroup functor with the following two properties: (1) τ is *inductive*, that is, for any group G, whenever $H \in \tau(G)$ is a p-group and $N \trianglelefteq G$, then $HN/N \in \tau(G/N)$; (2) τ is *hereditary*, that is, for any group G, whenever $H \in \tau(G)$ is a p-group and $H \leq E \leq G$, then $H \in \tau(E)$.

A class of groups $\mathfrak{F}$ is called *hereditary* (Mal'cev) if $H \in \mathfrak{X}$ whenever $H \leq G \in \mathfrak{X}$. In what follows, $\mathfrak{F}$ always denotes a non-empty hereditary saturated formation. Also, we use $\mathfrak{N}$ to denote the class of finite nilpotent groups, and $\mathfrak{U}$ to denote the class of finite supersoluble groups.

Following [10, Chap. III, Definition 3.1], a subgroup U of G is called $\mathfrak{F}$*-maximal* in G provided that (a) $U \in \mathfrak{F}$, and (b) if $U \leq V \leq G$ and $V \in \mathfrak{F}$, then $U = V$.

A chief factor L/K of G is said to be $\mathfrak{F}$-central (resp. $\mathfrak{F}$-eccentric) in G if $(L/K) \rtimes (G/C_G(L/K)) \in \mathfrak{F}$ (resp. $(L/K) \rtimes (G/C_G(L/K)) \notin \mathfrak{F}$). A normal subgroup N of G is called $\mathfrak{F}$-hypercentral in G if either $N = 1$ or every chief factor of G below N is $\mathfrak{F}$-central in G. The product of all $\mathfrak{F}$-hypercentral normal subgroups of G is denoted by $Z_{\mathfrak{F}}(G)$, and called the $\mathfrak{F}$-hypercenter of G. It is well known that if N is a normal $\mathfrak{F}$-hypercentral subgroup of G and for some subgroup A of G with $A \in \mathfrak{F}$, then $AN \in \mathfrak{F}$ (hence every $\mathfrak{F}$-hypercentral subgroup is contained in every $\mathfrak{F}$-maximal subgroup); moreover, if $G/N \in \mathfrak{F}$, then $G \in \mathfrak{F}$. Numerous applications of $\mathfrak{F}$-hypercentral subgroups, and in particular the subgroup $Z_{\mathfrak{F}}(G)$, are based on these two properties. Nevertheless, the condition "to be an $\mathfrak{F}$-hypercentral subgroup" is rather strong, so often it is desired to find and use "generalized" $\mathfrak{F}$-hypercentral subgroups having both these properties. For example, let $\mathrm{Int}_{\mathfrak{F}}(G)$ be the intersection of all $\mathfrak{F}$-maximal subgroups of G (we use here the notation in [23]), then every normal subgroup of G contained in $\mathrm{Int}_{\mathfrak{F}}(G)$ possesses both these properties, and all other general properties of the subgroup $\mathrm{Int}_{\mathfrak{F}}(G)$ are quite similar to the properties of the subgroup $Z_{\mathfrak{F}}(G)$ (see Lemma 2.1 below). This circumstance makes the subgroup $\mathrm{Int}_{\mathfrak{X}}(G)$ quite applicable and shows that the general problem of study of this one is actual. Some of such applications were already discussed in the papers [6,7,16,22,23].

In the present paper, we analyze some applications of the subgroup $\mathrm{Int}_{\mathfrak{U}}(G)$, that is, the intersection of all maximal supersoluble subgroups of G. In fact, we give necessary and sufficient conditions under which a normal subgroup of G is contained in $\mathrm{Int}_{\mathfrak{U}}(G)$.