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Nilpotent fusion systems [☆]

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ABSTRACT

Let p be a prime and let $\mathcal{F}$ be a saturated fusion system over a finite p-group P. A fusion system $\mathcal{F}$ is said to be nilpotent if $\mathcal{F} = \mathcal{F}_P(P)$. We give various criteria for a saturated fusion system $\mathcal{F}$ to be nilpotent, which generalize the analogous criteria for a finite group to be p-nilpotent.

1. Introduction

The notion of fusion system is originally due to L. Puig [34]. He used the term Frobenius category, but in this paper we use the term saturated fusion system following C. Broto, R. Levi and B. Oliver in [10, Definitions 1.1 and 1.2].

Let p be a prime and let G be a finite group with a Sylow p-subgroup P. The fusion system $\mathcal{F}_P(G)$ of the finite group G over P is the category whose objects are the sub-

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groups of P, and whose morphisms are given by $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{Hom}_G(Q,R)$ for $Q,R \leq P$.

The well-known Frobenius criterion on p-nilpotency for finite groups can be stated in terms of fusion systems: G is p-nilpotent if and only if $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

Frobenius' condition was introduced to p-blocks of finite groups by M. Broué and L. Puig in [11]. A block with this condition is called a nilpotent block. Roughly speaking, a block B of G with defect group D is said to be nilpotent if the block fusion system is $\mathcal{F}_D(D)$. A lot of interesting work on nilpotent blocks has been done by L. Puig, B. Külshammer and other authors, see [32,33,27] for example.

An abstract fusion system $\mathcal{F}$ over P is said to be nilpotent if $\mathcal{F} = \mathcal{F}_P(P)$. Fusion systems of p-nilpotent groups and nilpotent blocks are two important examples of nilpotent fusion systems. Moreover, in [12], the authors study the nilpotent p-local finite groups, in the sense that the underling fusion system is nilpotent. For more work on nilpotent fusion systems, see [12,13,18,26,29,31].

Let p be an odd prime. The p-nilpotency theorem of Glauberman and Thompson [19, Theorem 8.3.1] states that a finite group G is p-nilpotent if and only if $N_G(Z(J(P)))$ is p-nilpotent. The p-nilpotency theorem has been generalized to fusion systems, which says that $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $N_{\mathcal{F}}(Z(J(P))) = \mathcal{F}_P(P)$ by R. Kessar and M. Linckelmann in [26]. Moreover, the group $ZJ(P)$ can be replaced by a characteristic group $W(P)$ in [31] by S. Onofrei and R. Stancu.

In this paper, we focus on nilpotent fusion systems. Instead of Glauberman's characteristic subgroups (such as ZJ in [26] and W in [31]), we introduce $\Omega(P)$ and $\mathfrak{hnp}(\mathcal{F})$ to characterize the nilpotent fusion systems, where the notation is as described in Section 2 below. We give some necessary and sufficient conditions for a saturated fusion system to be nilpotent, which are analogous to some conditions for a group to be p-nilpotent. We prove most of the fusion analogues directly without resorting to reduction to finite groups to illuminate the reason why they have to be true. The direct proofs also provide new concise proofs of the group theoretic analogue.

An application of N. Itô's lemma [24, IV. 5.4] says that if every element of order p (resp. 2 or 4 if $p = 2$) lies in the center of G, then G is p-nilpotent, see [24, IV. 5.5]. This result still holds if we replace the center of G by the ith upper center of G [17, Corollary 6]. Inspired by this, we have Theorem 3.1. Then we derive the following theorem as a consequence.

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group P. Then $\mathcal{F} = \mathcal{F}_P(P)$ if and only if $Z_n(\mathcal{F}) = P$ for some $n \geq 1$.*

By this theorem, we have an equivalent definition of nilpotent fusion system: A fusion system $\mathcal{F}$ is nilpotent if $Z_n(\mathcal{F}) = P$ for some n. This definition is analogous to the definition of solvable fusion system. Recall that a fusion system $\mathcal{F}$ is solvable if $O_p^n(\mathcal{F}) = P$ for some n.