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Nonsolvable groups satisfying the prime-power hypothesis ☆

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A R T I C L E I N F O

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A B S T R A C T

Let G be a finite group with the degree set $\mathrm{cd}(G)$ of its complex irreducible characters. We call that G satisfies the prime-power hypothesis if, for distinct degrees $\chi(1), \psi(1) \in \mathrm{cd}(G)$, the greatest common divisor $\gcd(\chi(1), \psi(1))$ is a prime power. In this paper, we show that $|\mathrm{cd}(G)| \leqslant 18$ if G is a nonsolvable group satisfying the prime-power hypothesis.

1. Introduction

An interesting problem in character theory of finite groups is to determine which set of numbers can be a degree set of irreducible characters of some finite group. So far there is still no general criterion for any number set to be a degree set. With some additional assumption, it can be said about restrictions of the number of a degree set.

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Let G be a finite group and denote by $\operatorname{cd}(G)$ the degree set of complex irreducible characters of G. Suppose that any two of distinct degrees of irreducible characters of G are relatively prime. It is well known that $|\operatorname{cd}(G)| \leqslant 3$ if G is solvable and that $|\operatorname{cd}(G)| \leqslant 4$ for the general situation in terms of the results about the number of connected components of character degree graphs of finite groups (cf. [23]).

Weakening the coprime condition, M. Lewis initially investigated finite groups satisfying the one-prime hypothesis which means that, for distinct degrees $\chi(1), \psi(1) \in \operatorname{cd}(G)$, the greatest common divisor $\gcd(\chi(1), \psi(1))$ is 1 or a prime. He showed that $|\operatorname{cd}(G)| \leqslant 9$ for finite groups satisfying the one-prime hypothesis in a series of papers (cf. [17,18,20]).

Following this direction, M. Lewis introduced in [18] the concept of n-prime hypothesis. Specifically, G is said to satisfy the n-prime hypothesis if, for distinct degrees $\chi(1), \psi(1) \in \operatorname{cd}(G)$, the total number of (not necessarily different) primes of $\gcd(\chi(1), \psi(1))$ is at most n. In [18], the author conjectured that there is an integer valued function $f(n)$ defined for all nonnegative integer n such that $|\operatorname{cd}(G)| \leqslant f(n)$ whenever G is solvable. The conjecture is true for $n = 0, 1$ where $f(0) = 3$ and $f(1) = 9$. For $n = 2$, J. Hamblin and M. Lewis proved that $f(2) \leqslant 462{,}515$ (cf. [9,10]).

It seems natural to see what occurs if G is a nonsolvable group satisfying the 2-prime hypothesis. Based on Lewis' work about nonsolvable groups satisfying the one-prime hypothesis, we do investigate nonsolvable groups satisfying the prime-power hypothesis. Here the prime-power hypothesis means that, for distinct degrees $\chi(1), \psi(1) \in \operatorname{cd}(G)$, the greatest common divisor $\gcd(\chi(1), \psi(1))$ is a prime power. Our main result is the following:

Theorem 1.1. *Let G be a nonsolvable group satisfying the prime-power hypothesis. Then $|\operatorname{cd}(G)| \leqslant 18$.*

Obviously, the upper bound of Theorem 1.1 will not hold for solvable groups. Let G be a solvable group satisfying the prime-power hypothesis. Denote $n = \max\{\log_p(\gcd(\chi(1),\ \psi(1))) \mid \chi, \psi \in \operatorname{Irr}(G),\ \chi(1) \neq \psi(1),\ \text{and } p \mid |G|\}$. We believe that there is an explicit function of n for the exact upper bound of $|\operatorname{cd}(G)|$. But we prefer dealing with it in a successive paper. Combined with the work on finite groups satisfying the n-prime hypothesis for $n = 1, 2$, Theorem 1.1 gives a further trial to supplement the research of the highly focused degree graphs of finite groups.

The proof of Theorem 1.1 depends on the classification theorem of finite simple groups. After a case-by-case investigation, we can determine all of the possibilities of nonabelian chief factors of finite nonsolvable groups satisfying the prime-power hypothesis.

Theorem 1.2. *Let G be a finite group satisfying the prime-power hypothesis and M/N the nonabelian chief factor of G. Then M/N is one of the following groups in Table 1.*

In order to get our bound for $|\operatorname{cd}(G)|$, we need to further determine almost simple but not simple groups satisfying the prime-power hypothesis.