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Quasi-projective and quasi-liftable characters

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ABSTRACT

We study ordinary characters of a finite group G which vanish on the p-singular elements for a fixed prime p dividing the order of G. Such characters are called quasi-projective. We show that all quasi-projective characters of G are characters of projective modules if and only if the ordinary irreducible characters of G can be ordered in such a way that the top square fragment of the decomposition matrix is diagonal. Finally, we prove that the number of indecomposable quasi-projective characters of G is finite and characterize them in case of blocks with cyclic defect groups.

Dedicated to Otto Kegel on the occasion of his 80th birthday

1. Introduction

Throughout this paper let p always be a prime and let G be a finite group. The order of G is denoted by $|G|$, and $|G|_p$ stands for the p-part of $|G|$. For $n, m \in \mathbb{N}$ the notation $n \mid m$ means that n divides m. By $\mathrm{Irr}(G)$ we denote the set of ordinary irreducible characters of G and by $\mathrm{IBr}_p(G)$ that of irreducible p-Brauer characters with respect to a splitting p-modular system. We write Φ_φ for the ordinary character associated to the projective

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cover of the module corresponding to $\varphi \in \mathrm{IBr}_p(G)$. If χ is an ordinary character of G then χ° denotes the restriction of χ on the set of p-regular elements. Furthermore $(\cdot,\cdot)$ stands for the usual scalar product on the complex vector space of ordinary characters and $(\cdot,\cdot)^\circ$ means its restriction to the set of p-regular elements.

Definition 1.1. An ordinary character Λ of G is called quasi-projective if

$$\Lambda = \sum_{\varphi \in \mathrm{IBr}_p(G)} a_\varphi \Phi_\varphi \quad \text{with } a_\varphi \in \mathbb{Z}.$$

In case $a_\varphi \geq 0$ for all $\varphi \in \mathrm{IBr}_p(G)$ we call Λ projective, i.e., Λ is the ordinary character of a projective module.

Note that a character Λ is quasi-projective if and only if Λ vanishes on the set of p-singular elements of G (see [14, Theorem 2.13 and Corollary 2.17]). This shows that the coefficients a_φ are automatically integers since $a_\varphi = (\Lambda, \varphi)^\circ = (\Lambda, \varphi)$ and φ is an integer linear combination of ordinary irreducible characters restricted to p-regular elements (see [14, Corollary 2.16]). Although quasi-projective characters are a natural generalization of characters of projective modules, they did not attract much attention so far. In [15] the authors study quasi-projective characters of degree $|G|_p$, but only for Chevalley groups in defining characteristic p.

Definition 1.2. We call a p-Brauer character φ quasi-liftable if there exists an ordinary character χ such that $\chi^\circ = b\varphi$ with $b \in \mathbb{N}$.

Quasi-liftable irreducible Brauer characters φ for which $b > 1$ are of interest since they allow non-split self-extensions. More precisely, suppose that $\chi^\circ = b\varphi$ with $b > 1$. Let V be the module in characteristic p affording φ. According to a result of Thompson (see [6, Ch. I, Theorem 17.12]) there is a lattice affording χ whose Brauer reduction mod p is indecomposable. This implies in particular that there is an indecomposable module which is an extension of V by V, i.e., $\mathrm{Ext}^1_G(V, V) \neq 0$. One says that V has a self-extension. Surprisingly, for most Chevalley groups in defining characteristic $p > 3$ even liftable modules have self-extensions [16, Proposition 1.4]. As shown in [2] the self-extension phenomenon is very rare.

Quasi-liftable irreducible Brauer characters which are not liftable are hard to find, but they exist. Take for instance $G = 2.\mathrm{M}_{12}.2$, where M_{12} is the Mathieu group, and $p = 2$. Then there exists a non-liftable Brauer character $\varphi \in \mathrm{IBr}_2(G)$ of degree 44 and a $\chi \in \mathrm{Irr}(G)$ of degree 88 such that $\chi^\circ = 2\varphi$ (see decomposition matrices in [13]). Another example is provided by $G = ON.2$ again for $p = 2$. In this case there exists a $\chi \in \mathrm{Irr}(G)$ of degree 51 832 with $\chi^\circ = 2\varphi$ where $\varphi \in \mathrm{IBr}_2(G)$ is not liftable.

These examples may suggest that $p \mid b$ if $\chi^\circ = b\varphi$ for $\varphi \in \mathrm{IBr}_p(G)$ and $b > 1$. However this is not true. The group $G = {}^2F_4(2)'.2$ contains a non-liftable $\varphi \in \mathrm{IBr}_2(G)$ of degree 26 and characters $\chi, \psi \in \mathrm{Irr}(G)$ with $\chi^\circ = 2\varphi$ and $\psi^\circ = 3\varphi$. The principal 2-block