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A modular invariance property of multivariable trace functions for regular vertex operator algebras

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ABSTRACT

We prove an $\mathrm{SL}_2(\mathbb{Z})$-invariance property of multivariable trace functions on modules for a regular VOA. Applying this result, we provide a proof of the inversion transformation formula for Siegel theta series. As another application, we show that if V is a simple regular VOA containing a simple regular subVOA U whose commutant U^c is simple, regular, and satisfies $(U^c)^c = U$, then all simple U-modules appear in some simple V-module.

1. Introduction

The concept of a vertex operator algebra (VOA) was introduced by Borcherds [2] to explain a mysterious relation between the Monster simple group and the elliptic

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modular function $J(\tau)$. In the years since, this connection has been elucidated further and generalized to encompass a wide class of VOAs and elliptic modular forms. In the heart of this developing theory reside trace functions over modules of endomorphisms associated with the VOA. In particular, these functions include an operator formed from a matching of a distinguished element from the VOA, and a single variable in the complex upper half-plane. Meanwhile, the element resulting from this pairing resides in a one-dimensional Jordan subalgebra of the VOA, and begs the question whether trace functions exist which instead incorporate elements from larger Jordan subalgebras. The primary aim of this paper is to study such multivariable trace functions and establish functional equations for them with respect to the group $\mathrm{SL}_2(\mathbb{Z})$.

The development of these equations utilizes a seminal result of Zhu [12], which establishes that the space of trace functions on simple modules of a regular VOA is invariant under the standard action of $\mathrm{SL}_2(\mathbb{Z})$. In particular, Zhu shows that the action of an element of $\mathrm{SL}_2(\mathbb{Z})$ on a single-variable trace function on a simple module is a linear combination of the trace functions for all simple modules of the VOA with coefficients dependent on the representation of the element in $\mathrm{SL}_2(\mathbb{Z})$. As we lift Zhu's theory to the multivariable case below, we find that we recover these same coefficients. Using Verlinde's formula, we exploit this fact to show that every simple module of a regular subVOA whose commutant satisfies certain conditions is contained in a simple module of the VOA (see Theorem 2 below).

Beyond considering such regular subVOAs and their commutants, a number of important classes of VOAs are known to contain appropriate Jordan subalgebras and fit the framework presented here to construct multivariable trace functions. We discuss some of these below and look more closely at an application to lattice VOAs, where we formulate another proof of the transformation properties for Siegel theta functions. To explain our results in more detail, we first review the relevant theory and notation pertaining to VOAs.

A VOA is a quadruple $(V, Y(\cdot, z), \mathbf{1}, \omega)$, which we simply denote by V, consisting of a graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, a linear map $Y(\cdot, z) : V \to \mathrm{End}(V)[[z^{-1}, z]]$, and two notable elements $\mathbf{1} \in V_0$ and $\omega \in V_2$ called the Vacuum and Virasoro elements, respectively. We say v has weight n if $v \in V_n$ and denote the weight of v by $\mathrm{wt}(v)$ if it is not specified. An image $Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ of $v \in V$ is called a vertex operator of v, and it can be shown that $v_{\mathrm{wt}(v)-1}$ is a weight-preserving operator for a homogeneous element v. We denote this unique operator by $o(v)$ and extend it linearly. Meanwhile, the operators $L(n)$ defined by $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$ satisfy a Virasoro algebra bracket relation

$$[L(n), L(m)] = (n - m)L(n + m) + \delta_{n+m,0}\frac{n^3 - n}{12}c$$

for some $c \in \mathbb{C}$, called the central charge of V. The eigenvalues of $L(0)$ provide the weights on V, that is, $V_n = \{v \in V \mid L(0)v = nv\}$.