Journal of Algebra 444 (2015) 143–151

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Subgroup-closed lattice formations

Xiaolan Yi [a,*,1], S.F. Kamornikov [b]

[a] *Department of Mathematics, Zhejiang Sci-Tech University, Hangzhou 310018, PR China*

[b] *Gomel Branch of International University MITSO, Belarus, Gomel 246019, Belarus*

A R T I C L E I N F O

Article history:
Received 13 December 2014
Available online 26 August 2015
Communicated by E.I. Khukhro

MSC:
20D10
20D15
20D20

Keywords:
Finite group
Formation
Lattice
Residual

A B S T R A C T

A formation $\mathfrak{F}$ of finite groups is called a lattice formation if the set of all $\mathfrak{F}$-subnormal subgroups is a sublattice of the lattice of all subgroups in every finite group. The main result of this paper describes the family of all subgroup-closed lattice formations, and it can be regarded as an important step towards the solution of Shemetkov's classification problem.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group.

One of the most striking results in the theory of subnormal subgroups is the celebrated "join" theorem, proved by H. Wielandt in 1939 [12]: the subgroup generated by two

* Corresponding author.
E-mail addresses: yixiaolan2005@126.com (X. Yi), sfkamornikov@mail.ru (S.F. Kamornikov).

1 Research of the first author is supported by the NNSF grant of China (Grants 11101369; 11471055).

http://dx.doi.org/10.1016/j.jalgebra.2015.07.021

subnormal subgroups of a finite group is itself subnormal. As a result, the set $sn(G)$ of all subnormal subgroups of a group G is a sublattice of the subgroup lattice.

Wielandt's theorem was developed in formation theory using concepts of $\mathfrak{F}$-subnormality and K-$\mathfrak{F}$-subnormality. We refer to [2, Chapter 6] for a convenient account on the topic.

The first concept was proposed by R. Carter and T. Hawkes [3]. Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of a group G is said to be $\mathfrak{F}$-*subnormal* in G if either $H = G$ or there exists a maximal chain of subgroups

$$H = H_0 \subset H_1 \subset \cdots \subset H_n = G$$

such that $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. Following [2], the set of all $\mathfrak{F}$-subnormal subgroups of a group G is denoted by $sn_{\mathfrak{F}}(G)$.

It is rather clear that the $\mathfrak{N}$-subnormal subgroups of a group G for the formation $\mathfrak{N}$ of all nilpotent groups are subnormal, and they coincide in the soluble universe. However the equality $sn_{\mathfrak{N}}(G) = sn(G)$ does not hold in general.

To avoid the above situation, O.H. Kegel [7] introduced a somewhat different notion of $\mathfrak{F}$-subnormality. It unites the notions of subnormal and $\mathfrak{F}$-subnormal subgroup.

A subgroup H of a group G is called $\mathfrak{F}$-*subnormal in the sense of Kegel* (or simply K-$\mathfrak{F}$-*subnormal*) in G if there exists a chain of subgroups

$$H = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that H_{i-1} is either normal in H_i or $H_i^{\mathfrak{F}} \subseteq H_{i-1}$ for all $i = 1, \ldots, n$. We shall write $H \in sn_{K-\mathfrak{F}}(G)$ and denote by $sn_{K-\mathfrak{F}}(G)$ the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G.

Obviously, $sn_{K-\mathfrak{N}}(G) = sn(G)$ for every group G.

Let $\mathfrak{F}$ be a formation. One might wonder whether the set of $\mathfrak{F}$-subnormal subgroups of a group forms a sublattice of the subgroup lattice. As the Example 6.3.1 in [2] shows, the answer is in general negative.

Therefore the following question naturally arises:

Which are the formations $\mathfrak{F}$ for which the set $sn_{\mathfrak{F}}(G)$ is a sublattice of the subgroup lattice of G for every group G?

This question was first proposed by L.A. Shemetkov in his monograph [9, Problem 12] in 1978 and it appeared in the Kourovka Notebook [8, Problem 9.75] in 1984.

In 1992, A. Ballester-Bolinches, K. Doerk, and M.D. Perez-Ramos [1] gave the answer to that question in the soluble universe for subgroup-closed saturated formations. In 1993, A.F. Vasil'ev, S.F. Kamornikov, and V.N. Semenchuk [11] published the solution of Shemetkov's problem in the general finite universe for subgroup-closed saturated formations. As a significant progress, in 2002, A.F. Vasil'ev and the second author in [10] characterized the subgroup-closed lattice formations which are soluble.

In 1978, O.H. Kegel [7] showed that if $\mathfrak{F}$ is a subgroup-closed formation such that $\mathfrak{F}\mathfrak{F} = \mathfrak{F}$, then the set of all K-$\mathfrak{F}$-subnormal subgroups of a group G is a sublattice of the