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Arithmetic of seminormal weakly Krull monoids and domains ☆

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ABSTRACT

We study the arithmetic of seminormal v-noetherian weakly Krull monoids with nontrivial conductor which have finite class group and prime divisors in all classes. These monoids include seminormal orders in holomorphy rings in global fields. The crucial property of seminormality allows us to give precise arithmetical results analogous to the well-known results for Krull monoids having finite class group and prime divisors in each class. This allows us to show, for example, that unions of sets of lengths are intervals and to provide a characterization of half-factoriality.

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1. Introduction

Let R be a noetherian domain. Then every nonzero nonunit $a \in R$ can be written as a finite product of atoms (irreducible elements), say $a = u_1 \cdot \ldots \cdot u_k$. Such a product is called a factorization of a in R. The main goal of factorization theory is to describe the various phenomena of non-uniqueness of factorizations by suitable arithmetical invariants (such as sets of lengths and unions of sets of lengths) and to study their relationship with classical ring-theoretical parameters of the underlying domain. Given an element $a \in R$, the set $\mathsf{L}(a)$ of all possible factorization lengths $k \in \mathbb{N}$ is called the set of lengths of a. Since R is noetherian, $\mathsf{L}(a)$ is a finite subset of positive integers. For $k \in \mathbb{N}$, let $\mathcal{U}_k(R)$ denote the set of all $l \in \mathbb{N}$ with the following property: There are atoms u_i and v_j for indices $i \in [1, k]$ and $j \in [1, l]$ such that $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_l$. Thus $\mathcal{U}_k(R)$ is the union of all sets of lengths containing k. In particular, a domain R is said to be half-factorial if $|\mathsf{L}(a)| = 1$ for every nonzero nonunit $a \in R$ (equivalently, $\mathcal{U}_k(R) = \{k\}$ for all $k \in \mathbb{N}$). We note that if R is not half-factorial, then there exists $a \in R$ with $|\mathsf{L}(a)| > 1$ and hence $|\mathsf{L}(a^N)| > N$ for each positive integer $N \in \mathbb{N}$. Throughout this manuscript we will study, for a seminormal weakly Krull domain R, sets of lengths $\mathsf{L}(a)$ where $a \in R$ and unions of sets of lengths $\mathcal{U}_k(R)$ where $k \in \mathbb{N}$.

Within factorization theory, there are two main cases. The first — and by far the best understood — case is that of Krull domains. Recall that a noetherian domain is Krull if and only if it is integrally closed. Let R be a Krull domain. Then arithmetical phenomena depend only on the class group and on the distribution of prime divisors in the classes. In particular, R is factorial if and only if the class group is trivial. Let G denote the class group of R and let $G_P \subset G$ denote the set of classes containing prime divisors. Then there is a transfer homomorphism from R to the monoid $\mathcal{B}(G_P)$ of zero-sum sequences over G_P, which preserves sets of lengths and other invariants (see Section 4). This is the basis for a full variety of arithmetical finiteness results for Krull domains. If the class group is finite and every class contains a prime divisor, even more is known. Suppose this is the case; that is, $|G| < \infty$ and $G_P = G$. Then the natural transfer homomorphism provides a perfect link to additive group and number theory. In particular, from this transfer homomorphism one is able to establish not only finiteness results, but precise arithmetical results (see [42, Chapter 6] and [39] for an overview). We now demonstrate this by way of a few examples. The first such result — indeed one of the first results in this area of factorization theory, due to Carlitz 1960 — is a characterization of half-factoriality: the domain R is half-factorial if and only if $|G| \le 2$. Secondly, it is known that the unions of sets of lengths $\mathcal{U}_k(R)$ are not only finite (which is true even in many non-Krull settings), but they are finite intervals. Precise values of other arithmetical invariants such as elasticity and the catenary degree have been determined using only the structure of the class group G.

Much less is known in the non-Krull case. Suppose R is noetherian but not integrally closed. Apart from certain classes of semigroup rings, the most investigated class is that of C-domains. Let $\overline{R}$ denote the integral closure and let $\mathfrak{f} = (R : \overline{R})$ denote the