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On nilpotent Lie algebras of small breadth

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ABSTRACT

A Lie algebra L is said to be of breadth k if the maximal dimension of the images of left multiplication by elements of the algebra is k. In this paper we give characterization of finite dimensional nilpotent Lie algebras of breadth less than or equal to two. Furthermore, using these characterizations we determined the isomorphism classes of these algebras.

1. Introduction

Classification of algebraic objects is a central theme of mathematical research. The classification of finite dimensional complex simple Lie algebras due to Killing and Cartan is well known (cf. [5]). However, due to the existence of the vast number of finite dimensional nilpotent Lie algebras the classification problem has been formidable for this class. As a result, many researchers have made progress by classifying nilpotent Lie algebras satisfying certain conditions. Research in finite group theory has followed a similar path. Simple groups have been classified, but the large number of finite p-groups has

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led researchers to investigate p-groups with added conditions. One such condition is the breadth of a finite p-group [7,9]. The breadth $b(G)$ of a finite p-group G is defined as the size of the largest conjugacy class in G. Analogously the breadth $b(L)$ of a Lie algebra L is defined to be the maximum of the dimensions of the images of ad_x for all $x \in L$. In [9], Vaughan-Lee settled a long standing conjecture by showing that for a finite p-group G of breadth $b = b(G)$, $|G^2| \leq p^{(b(b+1)/2)}$. More recently, Parmeggiani and Stellmacher [8] gave characterizations of finite p-groups of breadth 1 and 2. However, so far there does not exist a classification of these finite p-groups.

In this paper we give characterizations for nilpotent Lie algebras of breadth 1 and 2. In particular, we show that a finite dimensional nilpotent Lie algebra is of breadth 1 if and only if its derived algebra is one dimensional. We also show that a finite dimensional nilpotent Lie algebra L has breadth 2 if and only if either the derived algebra of L has dimension 2 or the derived algebra and the central quotient both have dimension 3. These results parallel results in finite p-groups.

Finally we use our characterizations to classify finite dimensional nilpotent Lie algebras of breadth 1 and 2. We define a Lie algebra to be *pure* if it does not have abelian ideals as direct summands. Then we classify finite dimensional pure nilpotent Lie algebras of breadth one and two since abelian summands can be added harmlessly. In particular, we show that a finite dimensional pure nilpotent Lie algebra of breadth 1 is isomorphic to a Heisenberg Lie algebra. For a finite dimensional pure nilpotent Lie algebras L, the center is contained in the derived algebra. By our characterization result, the dimension of the derived algebra of a finite dimensional pure nilpotent Lie algebra L of breadth 2 is either 2 or 3. We determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with three dimensional derived algebra. We also determine the isomorphism classes of finite dimensional pure nilpotent Lie algebras of breadth two with two dimensional derived algebra and one dimensional center. For the remaining case where the derived algebra and center have dimension 2 each, we determine their isomorphism classes up to dimension 6. We hope that these classification results will lead to corresponding classification results in finite p-groups.

2. Basic properties of breadth for Lie algebras

Let L be a finite dimensional Lie algebra over field F and A be an ideal of L. In this paper we assume $\mathrm{char}(F) \neq 2$. For any $x \in L$ we define $b_A(x) = \mathrm{rank}(\mathrm{ad}_x|_A)$ and $b_A(L) = \max\{b_A(x) \mid x \in L\}$. Clearly $b_A(L) \leq \dim[A, L]$. We define the *breadth* of L to be $b(L) = b_L(L)$. We denote $b(x) = b_L(x)$ for all $x \in L$. Clearly $b_A(L) \leq b(L)$. It follows from the definition that L is abelian if and only if $b(L) = 0$. Let $Z(L)$ denote the center of L. The following results are easy to prove.

Proposition 2.1. $\dim(L/Z(L)) \geq b(L) + 1$.

Proposition 2.2. *Let $L = L_1 \oplus L_2$ be a finite dimensional direct sum Lie algebra. Then $b(L) = b(L_1) + b(L_2)$.*