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Hilbert polynomials of multigraded filtrations of ideals

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ABSTRACT

Hilbert functions and Hilbert polynomials of $\mathbb{Z}^s$-graded admissible filtrations of ideals $\{\mathcal{F}(\underline{n})\}_{\underline{n}\in\mathbb{Z}^s}$ such that $\lambda\left(\frac{R}{\mathcal{F}(\underline{n})}\right)$ is finite for all $\underline{n} \in \mathbb{Z}^s$ are studied. Conditions are provided for the Hilbert function $H_{\mathcal{F}}(\underline{n}) := \lambda(R/\mathcal{F}(\underline{n}))$ and the corresponding Hilbert polynomial $P_{\mathcal{F}}(\underline{n})$ to be equal for all $\underline{n} \in \mathbb{N}^s$. A formula for the difference $H_{\mathcal{F}}(\underline{n}) - P_{\mathcal{F}}(\underline{n})$ in terms of local cohomology of the extended Rees algebra of $\mathcal{F}$ is proved which is used to obtain sufficient linear relations analogous to the ones given by Huneke and Ooishi among coefficients of $P_{\mathcal{F}}(\underline{n})$ so that $H_{\mathcal{F}}(\underline{n}) = P_{\mathcal{F}}(\underline{n})$ for all $\underline{n} \in \mathbb{N}^s$. A theorem of Rees about joint reductions of the filtration $\{\overline{I^r J^s}\}_{r,s\in\mathbb{Z}}$ is generalised for admissible filtrations of ideals in two-dimensional Cohen–Macaulay local rings. Necessary and sufficient conditions are provided for the multi-Rees algebra of an admissible $\mathbb{Z}^2$-graded filtration $\mathcal{F}$ to be Cohen–Macaulay.

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1. Introduction

The objective of this paper is to understand links among joint reductions of multigraded filtrations of ideals, the coefficients of their Hilbert polynomials, local cohomology modules of Rees algebra and depths of various associated multigraded rings of filtrations. In order to explain the principal results proved in this paper, we recall a few definitions, results and set up notation.

Throughout this paper let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d with infinite residue field and $I_1, \ldots, I_s$ be $\mathfrak{m}$-primary ideals of R. For $s \geq 1$, we put $e = (1, \ldots, 1)$, $\underline{0} = (0, \ldots, 0) \in \mathbb{Z}^s$ and for all $i = 1, \ldots, s$, $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^s$ where 1 occurs at ith position. For $\underline{n} = (n_1, \ldots, n_s) \in \mathbb{Z}^s$, we write $\underline{I}^{\underline{n}} = I_1^{n_1} \cdots I_s^{n_s}$ and $\underline{n}^+ = (n_1^+, \ldots, n_s^+)$ where

$$n_i^+ = \begin{cases} n_i & \text{if } n_i > 0 \\ 0 & \text{if } n_i \leq 0. \end{cases}$$

For $s \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s$, put $|\alpha| = \alpha_1 + \cdots + \alpha_s$. Define $\underline{m} = (m_1, \ldots, m_s) \geq \underline{n} = (n_1, \ldots, n_s)$ if $m_i \geq n_i$ for all $i = 1, \ldots, s$. By "for all large $\underline{n}$" we mean $\underline{n} \in \mathbb{N}^s$ and $n_i \gg 0$ for all $i = 1, \ldots, s$.

Definition 1.1. A set of ideals $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ is called a $\mathbb{Z}^s$-graded $\underline{I} = (I_1, \ldots, I_s)$-**filtration** if for all $\underline{m}, \underline{n} \in \mathbb{Z}^s$, (i) $\underline{I}^{\underline{n}} \subseteq \mathcal{F}(\underline{n})$, (ii) $\mathcal{F}(\underline{n})\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n}+\underline{m})$ and (iii) if $\underline{m} \geq \underline{n}$, $\mathcal{F}(\underline{m}) \subseteq \mathcal{F}(\underline{n})$.

Two kinds of Rees rings encode information about filtrations of ideals. To define these, let $t_1, t_2, \ldots, t_s$ be indeterminates and $\underline{t}^{\underline{n}} = t_1{}^{n_1} \cdots t_s{}^{n_s}$. The $\mathbb{N}^s$-graded **Rees ring of** $\mathcal{F}$ is $\mathcal{R}(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{N}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$. The $\mathbb{Z}^s$-graded **extended Rees ring of** $\mathcal{F}$ is $\mathcal{R}'(\mathcal{F}) = \bigoplus_{\underline{n} \in \mathbb{Z}^s} \mathcal{F}(\underline{n})\underline{t}^{\underline{n}}$. For $\mathcal{F} = \{\underline{I}^{\underline{n}}\}_{\underline{n} \in \mathbb{Z}^s}$, we set $\mathcal{R}(\mathcal{F}) = \mathcal{R}(\underline{I})$ and $\mathcal{R}'(\mathcal{F}) = \mathcal{R}'(\underline{I})$.

Definition 1.2. A $\mathbb{Z}^s$-graded $\underline{I} = (I_1, \ldots, I_s)$-filtration $\mathcal{F} = \{\mathcal{F}(\underline{n})\}_{\underline{n} \in \mathbb{Z}^s}$ of ideals in R is called an $\underline{I} = (I_1, \ldots, I_s)$-**admissible filtration** if $\mathcal{F}(\underline{n}) = \mathcal{F}(\underline{n}^+)$ for all $\underline{n} \in \mathbb{Z}^s$ and $\mathcal{R}'(\mathcal{F})$ is finite $\mathcal{R}'(\underline{I})$-module.

For an R-module M of finite length, we write $\lambda_R(M)$ for length of M as an R-module. If the context is clear we simply write $\lambda(M)$. Let I be an $\mathfrak{m}$-primary ideal of R. In [31], P. Samuel showed that for $n \gg 0$, the **Hilbert function** $H_I(n) = \lambda\left(\frac{R}{I^n}\right)$ coincides with a polynomial

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I)$$

of degree d, called the **Hilbert polynomial** of I. The coefficients $e_i(I)$ for $i = 0, 1, \ldots, d$ are integers, called the **Hilbert coefficients** of I. The multiplicity of I, namely $e_0(I)$ is