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Cyclic complexes, Hall polynomials and simple Lie algebras ☆

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ABSTRACT

In this paper we study the category $C_m(\mathscr{P})$ of m-cyclic complexes over $\mathscr{P}$, where $\mathscr{P}$ is the category of projective modules over a finite dimensional hereditary algebra A, and describe almost split sequences in $C_m(\mathscr{P})$. This is applied to prove the existence of Hall polynomials in $C_m(\mathscr{P})$ when A is representation finite and $m \neq 1$. We further introduce the Hall algebra of $C_m(\mathscr{P})$ and its localization in the sense of Bridgeland. In the case when A is representation finite, we use Hall polynomials to define the generic Bridgeland–Hall algebra of A and show that it contains a subalgebra isomorphic to the integral form of the corresponding quantum enveloping algebra. This provides a construction of the simple Lie algebra associated with A.

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1. Introduction

The Ringel–Hall algebra $\mathfrak{H}_v(A)$ of a finite dimensional algebra A over a finite field $\mathbb{F}_q$ was introduced by Ringel [28] in 1990. By definition, the algebra $\mathfrak{H}_v(A)$ has a basis the isoclasses (isomorphism classes) of A-modules, and the structure constants are given by counting certain submodules. Ringel then showed that when A is hereditary and representation finite, $\mathfrak{H}_v(A)$ is isomorphic to the positive part of the corresponding quantum enveloping algebra. Later on, Green [13] obtained a comultiplication formula for Ringel–Hall algebras of hereditary algebras and extended Ringel's algebraic realization to arbitrary types.

When A is hereditary of finite representation type, Ringel [30] proved that the structure constants of the Ringel–Hall algebra $\mathfrak{H}_v(A)$ are actually polynomials in q, called Hall polynomials. By evaluating Hall polynomials at $q = 1$, it was shown in [29] that the degenerate Ringel–Hall algebra $\mathfrak{H}_1(A)$ is isomorphic to the positive part of the associated universal enveloping algebra. In particular, this gives a realization of nilpotent parts of the semisimple Lie algebra associated with A. Based on Ringel's idea, Peng and Xiao [25] obtained a realization of the whole semisimple Lie algebra in terms of the root category of A. It turns out that Ringel–Hall algebra approach provides a nice framework for the realization of quantum enveloping algebras and Kac–Moody Lie algebras, see, e.g., [28,30,13,29,21,22,26,34].

After Ringel's discovery, some efforts have been made in order to obtain the *whole* quantum group. For example, many authors have studied Hall algebras associated with triangulated categories; see [37,20,35,39]. Recently, Bridgeland [5] introduced the Hall algebra of 2-cyclic complexes of projective modules over a finite dimensional hereditary algebra A and proved that by taking localization and reduction, the resulting algebra admits a subalgebra isomorphic to the whole quantum enveloping algebra associated with A. If, moreover, A is representation finite, then the two algebras coincide.

The present paper mainly deals with the category $C_m(\mathscr{P})$ of m-cyclic complexes of projective modules over a finite dimensional hereditary algebra A (setting $C_0(\mathscr{P}) = C^b(\mathscr{P})$ by convention). We first describe almost split sequences in $C_m(\mathscr{P})$ in a way similar to that given in [33] for $C_1(\mathscr{P})$. This allows us to prove the existence of Hall polynomials in $C_m(\mathscr{P})$ when A is representation finite and $m \neq 1$. Second, we introduce the Hall algebra of m-cyclic complexes and, based on [5], prove that its localization is isomorphic to the tensor product of m-copies of the extended Ringel–Hall algebra of A. Finally, under the assumption that A is connected and representation finite, i.e., up to Morita equivalence, A is given by a (connected) valued Dynkin quiver $\vec{\Delta}$, we use Hall polynomials to define the generic Bridgeland–Hall algebra of $\vec{\Delta}$ and show that its degenerate form is isomorphic to the universal enveloping algebra of the simple Lie algebra $\mathfrak{g}_\Delta$ associated with the underlying diagram Δ. This provides a realization of the entire $\mathfrak{g}_\Delta$ in terms of 2-cyclic complexes.