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The character degree simplicial complex of a finite group

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ABSTRACT

The character degree graph $\Gamma(G)$ of a finite group G has long been studied as a means of understanding the structural properties of G. For example, a result of Manz and Pálfy states that the character degree graph of a finite solvable group has at most two connected components. In this paper, we introduce the character degree simplicial complex $\mathcal{G}(G)$ of a finite group G. We provide examples justifying the study of this simplicial complex as opposed to $\Gamma(G)$, and prove an analogue of Manz's Theorem on the number of connected components that is dependent upon the dimension of $\mathcal{G}(G)$.

1. Introduction

A valuable way of studying the structure of a finite group G is through the study of its irreducible characters. The value that the character takes on the identity of the group is called the *degree* of the character, and in fact, much can be said about the structure of the group by simply examining the irreducible character degrees of the

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group. We write $\mathrm{Irr}(G)$ for the set of irreducible characters of a group G and we write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.

Historically, two different graphs have been associated with the set $\mathrm{cd}(G)$. The first is called the *character degree graph* of $\mathrm{cd}(G)$, denoted by $\Gamma(G)$. The vertices of this graph are the members of the set $\mathrm{cd}(G) \setminus \{1\}$, and there is an edge connecting two vertices if the corresponding irreducible character degrees have a nontrivial common divisor. The second graph associated with $\mathrm{cd}(G)$ is the *prime vertex graph*, denoted $\Delta(G)$, which has the primes dividing some member of $\mathrm{cd}(G)$ as its vertices, and there is an edge between two vertices if there is a member of $\mathrm{cd}(G)$ divisible by the two associated primes.

A plethora of research has been conducted about the graph theoretic properties that must be satisfied by both $\Gamma(G)$ and $\Delta(G)$ when G is a finite group. Perhaps the first result in this area is the following theorem of Manz [5].

Theorem 1.1 *(Manz). Suppose that G is a finite solvable group. Then the number of connected components of $\Gamma(G)$ is at most 2.*

Theorem 1.1 demonstrates that not all sets of integers can appear as $\Gamma(G)$ when G is a finite solvable group. Pálfy showed independently in [6] that when G is a finite solvable group, $\Delta(G)$ also has at most two connected components. These results have inspired many works in this area that seek to classify other constraints that can be placed upon both $\Gamma(G)$ and $\Delta(G)$ when G is a finite solvable group; see [4] for a thorough description.

It has long been suggested that one should really consider the *character degree simplicial complex* and the *prime vertex simplicial complex* of a finite group G instead of studying $\Gamma(G)$ or $\Delta(G)$, yet this proposal has long been overlooked. In this paper, we introduce the character degree simplicial complex of a finite group G, denoted $\mathcal{G}(G)$, and we provide results demonstrating the necessity of this definition. We do not examine the prime vertex graph or the prime vertex simplicial complex any further in this work.

Although not entirely algebraic in nature, the novelty of this area of study requires a certain amount of topological background. We provide the topological necessities for this paper in Section 2. Subsections 2.1, 2.2, and 2.3 are purely topological, providing the definitions and machinery that will be necessary in order to study our applications. Subsection 2.4 introduces the common divisor and character degree simplicial complex in more detail, while also providing the applications of the first three subsections of Section 2 to our context. Subsection 2.5 discusses an analogue of Theorem 1.1 that becomes nontrivial with the introduction of the character degree simplicial complex.

In Section 3, we will investigate the analogue of Theorem 1.1 discussed in Subsection 2.5. More specifically, we will obtain a bound on the rank of $\Pi_1(\mathcal{G}(G))$ in terms of the dimension of $\mathcal{G}(G)$. The primary result of Section 3 is the following.

Theorem A. *Suppose G is a finite solvable group with $\mathcal{G}(G)$ connected and $\dim(\mathcal{G}(G)) = n$. Then $\mathrm{rk}(\Pi_1(\mathcal{G}(G))) \leq n^2 + n - 1$.*