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# Stability of locally CMFPD homologies under duality

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We consider bounded complexes $P_\bullet$ of finitely generated projective $A$-modules whose homologies have finite projective dimension and are locally Cohen–Macaulay. We give a necessary and sufficient condition so that its dual $P_\bullet^*$ also has the same property.



## 1. Introduction

Throughout this paper $A$ will denote a Cohen–Macaulay (CM) ring with $\dim A_{\mathfrak{m}} = d \; \forall \; \mathfrak{m} \in Max(A)$. Throughout, "CM" abbreviates "*Cohen–Macaulay*" and "FPD" abbreviates "*finite projective dimension*", which clarifies the title of the paper.

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To introduce the main results, in this paper, let $M$ denote a finitely generated $A$-module with $proj\dim(M) = r < \infty$. When $A$ is local, then $M$ is Cohen–Macaulay if and only if $grade(M) = r$. In this case, $Ext^i(M, A) = 0 \ \ \forall\ i \neq r$. However, even in the non-local case, grade of $M$ is defined as $grade(M) := \min\{i : Ext^i(M, A) \neq 0\}$ (see [6]). Let $\mathcal{B}$ denote the category of such finitely generated $A$-modules $M$, with $proj\dim(M) = grade(M)$. Then, $\forall\ M \in \mathcal{B}$, $Ext^i(M, A) = 0 \ \ \forall\ i \neq proj\dim(M)$.

Now suppose $P_\bullet$ is an object in the category $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ of finite complexes of finitely generated projective $A$-modules, with homologies in $\mathcal{B}$. In this paper, we give a necessary and sufficient condition, for such a complex $P_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$, so that its dual $P^*_\bullet$ is also in $Ch^b_{\mathcal{B}}(\mathcal{P}(A))$.

To further describe this condition let the homology $H_t(P_\bullet) \neq 0$, at degree $t$ and $\rho_t = proj\dim(H_t(P_\bullet))$. The homomorphism $H_t(P_\bullet) \longrightarrow \frac{P_t}{B_t}$, where $B_t = \partial_{t+1}(P_{t+1})$, induces a homomorphism

$$\iota_t : Ext^{\rho_t}\left(\frac{P_t}{B_t}, A\right) \longrightarrow Ext^{\rho_t}\left(H_t(P_\bullet), A\right)$$

The main Theorem 3.6 states that the dual $P^*_\bullet \in Ch^b_{\mathcal{B}}(\mathcal{P}(A))$ if and only if $\iota_t$ is an isomorphism, whenever $H_t(P_\bullet) \neq 0$.

The theorem immediately applies to complexes $P^*_\bullet \in Ch^b(\mathcal{P}(A))$ whose homologies are locally CMFPD (see Corollary 3.7). The theorem also applies to a number of interesting subcategories of $\mathcal{B}$, for which $\iota_t$ is already an isomorphism, for all such $t$. Among them are the categories $\mathcal{B}(n) = \{M \in \mathcal{B} : proj\dim(M) = n\}$. Note, $\mathcal{A} := \mathcal{B}(d)$ is the category of modules of finite length and finite projective dimension. The stability of $P_\bullet \in Ch^b_{\mathcal{A}}(\mathcal{P}(A))$, under duality is a theorem in [4].

We underscore that this paper is part of a wider study [4,5] of duality of subcategories of derived categories and their Witt groups. While we have particular interest in the setting of singular varieties and we also provide further insight into nonsingular varieties in these articles. Our interest in Witt theory stems from the introduction of Chow–Witt groups, [1] and developed by Fasel [3], as obstruction groups for projective modules to split off a free direct summand. The readers are referred to [4] for further introductory comments. To be more specific, one of the primary motivations behind this study has been to address the Witt theory, for non-regular (Cohen–Macaulay) schemes $X$, with $\dim X = d$, while this article addresses the duality aspect of the same. Let $\mathscr{V}(X)$ denote the category of locally free sheaves on $X$ and $D^b(\mathscr{V}(X))$ denote the derived category of finite complexes of locally free sheaves. Let $\mathbb{M}(X, d)$ denote the subcategory of $Coh(X)$ with finite length and finite $\mathscr{V}(X)$-dimension. (*For unexplained notations, readers are referred to* [5,4].) As a consequence of Theorem 3.6, it follows that $D^b_{\mathbb{M}(X,d)}(\mathscr{V}(X))$ is closed under the usual duality induced by $\mathcal{E} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{O}_X)$. This allows us [5,4] to give a definition of shifted Witt groups $W^r(D^b_{\mathbb{M}(X,d)}(X))$ of $D^b_{\mathbb{M}(X,d)}(X)$, while $D^b_{\mathbb{M}(X,d)}(X)$ fails to inherit a triangulated structure. Also note that $\mathbb{M}(X, d)$ has a duality sending $\mathcal{F} \mapsto \mathcal{E}xt^d(\mathcal{F}, \mathcal{O}_X)$ and hence a Witt group $W(\mathbb{M}(X, d))$ is defined. It was established