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Identities, Ore extensions and character skew derivations

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ABSTRACT

Following [17], a σ-derivation δ is called *quasi*-algebraic if there exist $a, a_i \in Q$ such that $\delta^k(x) + \sum_{i=1}^{k-1} a_i\delta^i(x) + ax - \sigma^k(x)a = 0$ for $x \in Q$. Let k be the minimal such integer and $g_0, g_1, \ldots$ mutually outer automorphisms of Q. Let $f(z_{ij})$ be a generalized polynomial in variables z_{ij}, where $i \geq 0$ and $0 \leq j < k$. It is shown in [6] that if $f(g_i\delta^j(x))$ or $f(\delta^j g_i(x))$ is an identity, then so is $f(z_{ij})$. We generalize this to sets of skew derivations in Theorems 1 and 2. In Theorem 3, we apply these results to the Ore extensions of several variables of [2] and in Theorem 4 to the character Hopf algebras of the character skew derivations of [12].

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1. Results

A map $\theta : R \to R$ is called *additive* if $\theta(x+y) = \theta(x) + \theta(y)$ for $x, y \in R$. The set of all additive maps $R \to R$ forms a ring under point-wise addition and composition multiplication. By an automorphism of R, we mean a *bijective* additive map $\sigma : R \to R$ such that $\sigma(xy) = \sigma(x)\sigma(y)$ for $x, y \in R$. Given a unit $u \in R$, the map $\mathrm{I}_u : x \in R \mapsto uxu^{-1}$ defines an automorphism of R, called the inner automorphism defined by u. By a σ-derivation of R, where σ is an automorphism of R, we mean an additive map $\delta : R \to R$ such that

$$\delta(xy) = \delta(x)y + \sigma(x)\delta(y) \quad \text{for } x, y \in R.$$

We call σ the associated automorphism of δ. For brevity, we also call δ a *skew* derivation by suppressing σ. Given $a \in R$, the map $\mathrm{ad}_\sigma(a) : x \in R \mapsto ax - \sigma(x)a$ defines a σ-derivation of R, called the inner σ-derivation defined by a. By a differential polynomial φ, we mean a generalized polynomial with its variables acted by composition products of automorphisms and skew derivations of R. (See [8–11].) We call φ *linear* in the variable x if $\varphi = \sum_i A_i \mu_i(x) B_i$, where A_i, B_i are differential polynomials containing no x and where μ_i are composition products of automorphisms and skew derivations of R. We call φ *multi-linear* if it is linear in all the variables it contains. We call φ *linear* it contains only one variable and is linear in this variable. If a differential polynomial φ vanishes for any evaluations of its variables in R, then we call φ a differential *identity* of R. An identity φ is called linear or multi-linear respectively if the differential polynomial φ is linear or multi-linear respectively. Our aim here is to investigate these identities. Let Q denote the left Martindale quotient ring of R. (See [1,18] for a definition.) As shown in [7–11], we have to work in Q as follows: Let $I \trianglelefteq R$ denote that I is a two-sided ideal of Q. For each $a \in Q$, the family $\{a + I : 0 \neq I \trianglelefteq R\}$ forms a neighborhood system of Q at the point a by the primeness of R. These neighborhood systems define a topology, called the *ideal* topology of Q. Clearly, an additive map $\theta : Q \to Q$ is continuous if and only if for any two-sided ideal $0 \neq I \trianglelefteq R$ there exists a two-sided ideal $0 \neq J \trianglelefteq R$ such that $\theta(J) \subseteq I$. An automorphism σ of Q is called *bi-continuous* if both σ and σ^{-1} are continuous. A skew derivation is called continuous if it is continuous and its associated automorphism is bi-continuous. Any automorphism of R can be extended uniquely to a bi-continuous automorphism of Q. Analogously, any skew derivation of R, together with its associated automorphism, can be extended uniquely to a continuous skew derivation of Q. So any differential polynomial of R extends to a unique differential polynomial of Q. By [22], any linear differential identity of R extends to a linear differential identity of Q in this way. It thus suffices to investigate linear differential identities of Q involving bi-continuous automorphisms and continuous skew derivations. So we set

$$\mathbb{A} := \text{the set of bi-continuous automorphisms of } Q,$$

$$\mathbb{L}_\sigma := \text{the set of continuous } \sigma\text{-derivations of } Q \quad \text{for each } \sigma \in \mathbb{A} \quad \text{and}$$