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Simple toroidal vertex algebras and their irreducible modules

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ABSTRACT

In this paper, we continue the study on toroidal vertex algebras initiated in [15], to study concrete toroidal vertex algebras associated to toroidal Lie algebra $L_r(\hat{\mathfrak{g}}) = \hat{\mathfrak{g}} \otimes L_r$, where $\hat{\mathfrak{g}}$ is an untwisted affine Lie algebra and $L_r = \mathbb{C}\left[t_1^{\pm 1}, \dots, t_r^{\pm 1}\right]$. We first construct an $(r+1)$-toroidal vertex algebra $V(T,0)$ and show that the category of restricted $L_r(\hat{\mathfrak{g}})$-modules is canonically isomorphic to that of $V(T,0)$-modules. Let $\mathfrak{c}$ denote the standard central element of $\hat{\mathfrak{g}}$ and set $S_{\mathfrak{c}} = U(L_r(\mathbb{C}\mathfrak{c}))$. We furthermore study a distinguished subalgebra of $V(T,0)$, denoted by $V(S_{\mathfrak{c}},0)$. We show that (graded) simple quotient toroidal vertex algebras of $V(S_{\mathfrak{c}},0)$ are parametrized by a $\mathbb{Z}^r$-graded ring homomorphism $\psi : S_{\mathfrak{c}} \to L_r$ such that $\mathrm{Im}\psi$ is a $\mathbb{Z}^r$-graded simple $S_{\mathfrak{c}}$-module. Denote by $L(\psi,0)$ the simple quotient $(r+1)$-toroidal vertex algebra of $V(S_{\mathfrak{c}},0)$ associated to ψ. We determine for which ψ, $L(\psi,0)$ is an integrable $L_r(\hat{\mathfrak{g}})$-module and we then classify irreducible $L(\psi,0)$-modules for such a ψ. For our need, we also obtain various general results.

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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra equipped with the normalized Killing form $\langle \cdot, \cdot \rangle$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}\left[t_0^{\pm 1}\right] \oplus \mathbb{C}\mathfrak{c}$ be the untwisted affine Lie algebra. It is well-known (see [7,13]) that there exists a canonical vertex algebra $V_{\hat{\mathfrak{g}}}(\ell, 0)$ associated to $\hat{\mathfrak{g}}$ for each $\ell \in \mathbb{C}$ and the category of $V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules is canonically isomorphic to the category of restricted $\hat{\mathfrak{g}}$-modules of level ℓ. Denote by $L_{\hat{\mathfrak{g}}}(\ell, 0)$ the unique graded simple quotient vertex algebra of $V_{\hat{\mathfrak{g}}}(\ell, 0)$. It was known (see [10]) that $L_{\hat{\mathfrak{g}}}(\ell, 0)$ is an integrable $\hat{\mathfrak{g}}$-module if and only if ℓ is a non-negative integer. Furthermore, it was known (see [7,3,13,17, 18,4]) that if ℓ is a non-negative integer, the category of $L_{\hat{\mathfrak{g}}}(\ell, 0)$-modules is naturally isomorphic to the category of restricted integrable $\hat{\mathfrak{g}}$-modules of level ℓ.

Toroidal Lie algebras, which are essentially central extensions of multi-loop Lie algebras, generalizing affine Kac–Moody Lie algebras, form a special family of infinite dimensional Lie algebras closely related to extended affine Lie algebras (see [1]). A natural connection of toroidal Lie algebras with vertex algebras has also been known (see [2]), which uses one-variable generating functions for toroidal Lie algebras. By considering multi-variable generating functions (cf. [8,9]), toroidal vertex algebras were introduced in [15], which generalize vertex algebras in a certain natural way.

The essence of an $(r+1)$-toroidal vertex algebra V is that to each vector $v \in V$, a multi-variable vertex operator $Y(v; x_0, \mathbf{x})$ is associated, which satisfies a Jacobi identity. It is important to note that for a vertex algebra $(V, Y, \mathbf{1})$, the so-called creation property states that

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad (Y(v, x)\mathbf{1})|_{x=0} = v \qquad \text{for } v \in V,$$

which implies that V as a V-module is cyclic on the vacuum vector $\mathbf{1}$ and the vertex operator map $Y(\cdot, x)$ is always injective. However, this is *not* the case for an $(r+1)$-toroidal vertex algebra in general. For an $(r+1)$-toroidal vertex algebra V, denote by V^0 the submodule of the adjoint module V generated by $\mathbf{1}$, which is an $(r+1)$-toroidal vertex subalgebra. It was proved in [15] that V^0 has a canonical vertex algebra structure. To a certain extent, V^0 to V is the same as the core subalgebra to an extended affine Lie algebra (see [1]). In this paper, we explore V^0 more in various directions. In particular, we show that V^0 is a vertex $\mathbb{Z}^r$-graded algebra in a certain sense (see Section 3 for the definition). It is proved that if V is a simple $(r+1)$-toroidal vertex algebra, then V^0 is also a simple $(r+1)$-toroidal vertex algebra. Let L be any quotient $(r+1)$-toroidal vertex algebra of V. It is proved (see Proposition 2.26) that a V-module W is naturally an L-module if and only if W is naturally an L^0-module.

In this paper, we also study $(r+1)$-toroidal vertex algebras naturally arisen from toroidal Lie algebras. Specifically, we consider Lie algebra

$$\tau = \hat{\mathfrak{g}} \otimes \mathbb{C}\left[t_1^{\pm 1}, \ldots, t_r^{\pm 1}\right], \tag{1.1}$$