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Quasi-hereditary structure of twisted split category algebras revisited

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A B S T R A C T

Let k be a field of characteristic 0, let C be a finite split category, let α be a 2-cocycle of C with values in the multiplicative group of k, and consider the resulting twisted category algebra $A := k_\alpha\mathsf{C}$. Several interesting algebras arise that way, for instance, the Brauer algebra. Moreover, the category of biset functors over k is equivalent to a module category over a condensed algebra $\varepsilon A\varepsilon$, for an idempotent ε of A. In [2] the authors proved that A is quasi-hereditary (with respect to an explicit partial order $\leqslant$ on the set of irreducible modules), and standard modules were given explicitly. Here, we improve the partial order $\leqslant$ by introducing a coarser order $\trianglelefteq$ leading to the same results on A, but which allows to pass the quasi-heredity result to the condensed algebra $\varepsilon A\varepsilon$ describing biset functors, thereby giving a different proof of a quasi-heredity result of Webb, see [21]. The new partial order $\trianglelefteq$ has not been considered before, even in the special cases, and we evaluate it explicitly for the case of biset functors and the Brauer algebra. It also puts further restrictions on the possible composition factors of standard modules.

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1. Introduction

Suppose that k is a field and that C is a finite category, that is, the morphisms in C form a finite set. Suppose further that α is a 2-cocycle of C with values in $k^\times$. Then the *twisted category algebra* $k_\alpha\mathsf{C}$ is the finite-dimensional k-algebra with the morphisms in C as a k-basis, and with multiplication induced by composition of morphisms, twisted by the 2-cocycle α; for a precise definition, see 3.1. In the case where the category has one object only this just recovers the notion of a twisted monoid algebra.

In recent years, (twisted) category algebras and (twisted) monoid algebras have been intensively studied by B. Steinberg et al., for instance in [10], as well as by Linckelmann and Stolorz, who, in particular, determined the isomorphism classes of simple $k_\alpha\mathsf{C}$-modules in [17]. As a consequence of [17, Theorem 1.2] the isomorphism classes of simple $k_\alpha\mathsf{C}$-modules can be parametrized by a set Λ of pairs whose first entry varies over certain finite groups related to C (called *maximal subgroups* of C), and whose second entry varies over the isomorphism classes of simple modules over twisted group algebras of these maximal subgroups.

For convenience, in the following we shall suppose that k has characteristic 0, but this condition can be relieved, as we shall see in Theorem 4.3. Moreover, we suppose that the category C is *split*, that is, every morphism in C is a split morphism, see 3.1(a). It has been shown by the authors in [2] and, independently, by Linckelmann and Stolorz in [18] that the resulting twisted category algebra $k_\alpha\mathsf{C}$ is quasi-hereditary in the sense of [6]. In [2] we also determined the standard modules of $k_\alpha\mathsf{C}$ with respect to a natural partial order $\leq$ on the labelling set Λ of isomorphism classes of simple modules, which depends only on the first entries of pairs in Λ and is explained in 3.6.

Since $k_\alpha\mathsf{C}$ is quasi-hereditary with respect to $(\Lambda, \leq)$, it is also quasi-hereditary with respect to any refinement of $\leq$, and the corresponding standard and costandard modules are the same as those with respect to $(\Lambda, \leq)$. This is an immediate consequence of Definition 2.1 below.

In this paper we introduce a new partial order $\trianglelefteq$ on Λ such that the partial order $\leq$ is a refinement of $\trianglelefteq$. We shall then show in Theorem 4.3 that the algebra $k_\alpha\mathsf{C}$ remains quasi-hereditary with respect to this new partial order. Furthermore, we shall show that the standard and costandard modules of $k_\alpha\mathsf{C}$ with respect to the two partial orders coincide, and we shall give explicit descriptions of these modules. The partial order $\trianglelefteq$ seems more natural than the initial one, since it depends on both entries of the pairs in Λ, and it allows to pass the hereditary structure to idempotent condensed algebras $\varepsilon \cdot k_\alpha\mathsf{C} \cdot \varepsilon$, for $\varepsilon^2 = \varepsilon \in k_\alpha\mathsf{C}$, in a particular case we are interested in and that is related to the category of biset functors, see Section 7. In Sections 5 and 6 we shall give a number of possible reformulations and simplifications of the defining properties of the partial order $\trianglelefteq$ that are particularly useful when considering concrete examples.

It is known, by work of Wilcox [22], that diagram algebras such as Brauer algebras, Temperley–Lieb algebras, partition algebras, and relatives of these arise naturally as twisted split category algebras and twisted regular monoid algebras; for a list of