Journal of Algebra 440 (2015) 545–593

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Witt vector rings and the relative de Rham Witt complex ☆,☆☆

Joachim Cuntz, Christopher Deninger *

A R T I C L E I N F O

Article history:
Received 18 May 2015
Available online 13 July 2015
Communicated by V. Srinivas

Keywords:
Witt vectors
de Rham Witt complex

A B S T R A C T

In this paper we develop a novel approach to Witt vector rings and to the (relative) de Rham Witt complex. We do this in the generality of arbitrary commutative algebras and arbitrary truncation sets. In our construction of Witt vector rings the ring structure is obvious and there is no need for universal polynomials. Moreover a natural generalization of the construction easily leads to the relative de Rham Witt complex. Our approach is based on the use of free or at least torsion free presentations of a given commutative ring R and it is an important fact that the resulting objects are independent of all choices. The approach via presentations also sheds new light on our previous description of the ring of p-typical Witt vectors of a perfect $\mathbb{F}_p$-algebra as a completion of a semigroup algebra. We develop this description in different directions. For example, we show that the semigroup algebra can be replaced by any free presentation of R equipped with a linear lift of the Frobenius automorphism. Using the result in Appendix A by Umberto Zannier we also extend the description of the Witt vector ring as a completion to all $\overline{\mathbb{F}}_p$-algebras with injective Frobenius map.

☆ With an appendix by Umberto Zannier.

☆☆ Research supported by DFG through CRC 878, by ERC through AdG 267079 and by the MPIM Bonn.

* Corresponding author.
E-mail address: c.deninger@uni-muenster.de (C. Deninger).

http://dx.doi.org/10.1016/j.jalgebra.2015.05.029
0021-8693/

Introduction

Motivated by the problem of constructing field extensions of degree p^n, Witt introduced the ring of Witt vectors of a given ring. In his ingenious construction the ring operations are defined using infinite sequences of universal polynomials. Using Witt vector rings one may for example pass from perfect fields k of positive characteristic to unramified complete discrete valuation rings R with residue field k and quotient field of characteristic zero. By now, Witt vector rings and their more general variants are a classical tool in many branches of mathematics ranging from algebra and algebraic number theory over arithmetic geometry to homotopy theory. However the construction of the ring of Witt vectors and, even more so, the construction of the associated de Rham–Witt complex is still nowadays considered to be not so easy.

In this paper we develop a novel approach to Witt vector rings and to the (relative) de Rham Witt complex. We do this in the generality of arbitrary commutative algebras and arbitrary truncation sets. In particular the big and the p-isotypical theories are covered. In our construction of Witt vector rings the ring structure is obvious and there is no need for universal polynomials. Moreover a natural generalization of the construction easily leads to the relative de Rham Witt complex.

Our approach is based on the use of free or at least torsion free presentations of a given commutative ring R and it is an important fact that the resulting objects are independent of all choices. The approach via presentations also sheds new light on the description in [2] of the ring of p-typical Witt vectors of a perfect $\mathbb{F}_p$-algebra as a completion of a semigroup algebra. We develop this description in different directions. For example, we show that the semigroup algebra can be replaced by any free presentation of R equipped with a linear lift of the Frobenius automorphism. Using the result in Appendix A by Umberto Zannier we also extend the description of the Witt vector ring as a completion to all $\overline{\mathbb{F}}_p$-algebras with injective Frobenius map.

The paper is self-contained and elementary. Except for the results comparing the objects that we obtain to the conventional ones, it can be read without knowing the classical theories in the title. For these we refer to [16,10,15,9,17,18,14,8,1] and the appendix of [13] as far as Witt vector rings are concerned. For the de Rham Witt theory we mention the works [3,11,8,12] and the references given there. More advanced topics are discussed in [4–6] for example.

Our constructions of Witt vector rings and of the de Rham Witt complex are so simple that it is worthwhile to give the details in this introduction — at least in the p-typical case and for $\mathbb{Z}$-algebras R. In case R is an $\mathbb{F}_p$-algebra this gives a quick alternative approach to the original de Rham Witt complex in [11]. We will describe the construction of a commutative ring $E(R)$ and a differential graded ring $E\Omega^{\bullet}_{R/\mathbb{Z}}$ with $E\Omega^0_{R/\mathbb{Z}} = E(R)$ and equipped with Frobenius and Verschiebung maps. The ring $E(R)$ is isomorphic to the p-typical Witt ring $W(R)$ and there is a canonical isomorphism:

$$E\Omega^{\bullet}_{R/\mathbb{Z}} \cong W\Omega^{\bullet}_{R/\mathbb{Z}} := \varprojlim_n W_n\Omega^{\bullet}_{R/\mathbb{Z}} \,. \tag{1}$$