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Methods for parametrizing varieties of Lie algebras

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ABSTRACT

A real n-dimensional anticommutative nonassociative algebra is represented by an element of $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. For each $\mu \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, there is a unique subset $\Lambda \subseteq \{(i,j,k) : 1 \leq i < j \leq n, 1 \leq k \leq n\}$ so that the structure constant μ_{ij}^k with $i < j$ is nonzero if and only if $(i,j,k) \in \Lambda$. The set of all $\mu \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ with subset Λ is denoted $\mathcal{S}_\Lambda(\mathbb{R})$; these sets stratify $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We describe how to smoothly parametrize the Lie algebras in $\mathcal{S}_\Lambda(\mathbb{R})$ up to isomorphism, for Λ satisfying certain frequently seen hypotheses.

1. Introduction

In many areas of mathematics, it is natural to subdivide an object into smaller, simpler pieces and to parametrize each piece in some useful, controlled way (Yomdin [22]). A *semi-algebraic set* is a subset of $\mathbb{R}^n$ defined by a finite number of polynomial equations and inequalities, along with the operations of intersection and union. Every closed and bounded semi-algebraic set is semi-algebraically triangulable (Benedetti and Risler [3]). A *parametrization* of semi-algebraic subset A of $\mathbb{R}^n$ is a collection of semi-algebraic subsets A_j that cover A, along with surjective *charts* $\varphi_j : I^{n_j} \to A_j$, where I^{n_j} is a cube

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in $\mathbb{R}^{n_j}$, such that each chart φ_j is algebraic and is a homeomorphism from the interior of I^{n_j} to the interior of A_j.

Here we are interested in parametrizing varieties of Lie algebras. These sets are not semi-algebraic; rather, they are quotients of semi-algebraic sets by the action of a Lie group. Let A be a semi-algebraic subset of $\mathbb{R}^n$ and let $A/\sim$ be the quotient of A under the action of a group. Let $\{A_j\}$ be a collection of invariant semi-algebraic subsets that cover A. A *parametrization* of $A/\sim$ is the collection $\{A_j/\sim\}$ of semi-algebraic quotients, along with *charts* $\varphi_j : I^{n_j} \to A_j/\sim$, such that each chart φ_j is algebraic and is a homeomorphism from the interior of I^{n_j} to the interior of $A_j/\sim$. Our goal is to find explicit parametrizations of specific subsets of the variety $\widetilde{\mathcal{N}}_n(\mathbb{R})$ of real nilpotent Lie algebras of dimension n.

Real nilpotent Lie algebras of dimension 7 and lower have been classified (Morozov [13], Seeley [20], Gong [8]). For $n \leq 6$, $\widetilde{\mathcal{N}}_n(\mathbb{R})$ is discrete. The space $\widetilde{\mathcal{N}}_7(\mathbb{R})$ is the union of isolated points and one-parameter families. In dimension eight and higher, nilpotent Lie algebras have not been classified, and there are many components of $\widetilde{\mathcal{N}}_n(\mathbb{R})$ of higher dimension (Ancochea-Bermúdez et al. [1]). Furthermore, there are large families of “characteristically nilpotent” Lie algebras (Hakimjanov [10]); these do not admit nontrivial semisimple derivations. Due to these complications, in dimension eight and higher, instead of analyzing all of $\widetilde{\mathcal{N}}_n(\mathbb{R})$, it is natural to focus on a tractable subset $\mathcal{R}$ of $\widetilde{\mathcal{N}}_n(\mathbb{R})$. For example, $\mathcal{R}$ might be

- the set of two-step nilpotent Lie algebras,
- the set of filiform or quasi-filiform Lie algebras,
- the set of $\mathbb{N}$-graded or naturally graded nilpotent Lie algebras,
- a set of nilpotent Lie algebras that admit a special kind of structure (affine, symplectic, contact, almost-complex, Kähler, etc.).

Or, $\mathcal{R}$ might be defined as the intersection of two or more such sets.

The subclass of filiform Lie algebras is not discrete in dimensions seven and higher and has been the setting for many kinds of classification problems (See, for example, Vergne [21], Millionschikov [12], Burde [5], Arroyo [2]). The classification of complex two-step nilpotent Lie algebras of dimension 9 and lower was completed in Galitski and Timashev [7], and dimensions of moduli spaces of two-step nilpotent algebras of types (p, q) were found in Eberlein [6].

Another subclass of nilpotent Lie algebras is the set of nilpotent Lie algebras whose “Nikolayevsky derivation” has positive eigenvalues all of multiplicity one. (See Nikolayevsky [14] for a definition of the Nikolayevsky derivation.) A subset of this subclass in dimensions 7 and 8 was classified in Kadioglu and Payne [11]. In fact, the motivation for this work is to develop the necessary tools for the completion in Payne [18] of the classification begun in Kadioglu and Payne [11]. In Payne [18], we classify all nilpotent Lie algebras of dimensions 7 and 8 for which the Nikolayevsky derivation is simple with positive eigenvalues. The methods developed here are essential in Payne [18] for the de-