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Generators for the representation rings of certain wreath products

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ABSTRACT

Working in the setting of Deligne categories, we generalize a result of Marin that hooks generate the representation ring of symmetric groups to wreath products of symmetric groups with a fixed finite group or Hopf algebra. In particular, when we take the finite group to be cyclic order 2 we recover a conjecture of Marin about Coxeter groups in type B.

1. Introduction

In [3], Deligne defined the categories $\underline{Rep}(S_t)$ for t an arbitrary complex number. In the context of the Church–Farb framework of representation stability [1] we may think of these Deligne categories at generic values of t as models for stable categories of representations of the symmetric group. In particular they satisfy the following "stable" properties:

- For generic t, $\underline{Rep}(S_t)$ is semisimple with irreducible objects $\tilde{V}(\lambda)$ indexed by partitions. These interpolate the irreducible representations $V(\lambda(n))$ of S_n with $n \gg 0$,

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where we add a sufficiently long first row to λ to make it the right size. In particular these representations are known to have polynomial growth of dimension $Dim(V(\lambda(n))) = p_\lambda(n)$, and in the Deligne category we have $Dim(\tilde{V}(\lambda)) = p_\lambda(t)$.

- If $k \in \mathbb{Z}_{>0}$ we have induction functors $Rep(S_k) \boxtimes \underline{Rep}(S_t) \to \underline{Rep}(S_{t+k})$, where $Rep(S_k)$ denotes the usual category of complex representations of S_k. The multiplicity $\tilde{c}^{\nu}_{\lambda,\mu}$ of $\tilde{V}(\nu)$ in $\mathrm{Ind}(V(\lambda) \boxtimes \tilde{V}(\mu))$ is equal to the stable limit of Littlewood–Richardson coefficients $c^{\nu(n+k)}_{\lambda,\mu(n)}$. Similar statements hold for restriction, with an appropriate version of Frobenius reciprocity.
- The structure constants for the tensor product are the so called reduced Kronecker coefficients which are the stable limits of the Kronecker coefficients.

In [9] Marin proves that hooks, i.e. partitions of the form $(n-k, 1^k)$ generate the representation ring of $\mathrm{Rep}(S_n)$. While he doesn't use the language, his proof mostly takes place in the stable setting and his argument shows that hooks freely generate the stable representation ring. This then implies they must generate in the classical setting (although not freely). So we may think of this result as an application of stable representation theory to classical representation theory.

In the Deligne category setting Marin's result appears in Deligne's original paper [3], saying that the Grothendieck ring of the Deligne category is freely generated by objects corresponding to hooks. The result for the classical case follows by projecting from the Deligne category onto $\mathrm{Rep}(S_n)$, and looking at the induced map of Grothendieck rings.

The proof is done by defining a filtration on the Deligne category such that the associated graded Grothendieck ring is isomorphic in a natural way to the ring $(\bigoplus_n K_0(\mathrm{Rep}(S_n)), \cdot)$ with multiplication coming from inducing representations from $S_n \times S_m$ to S_{n+m}. This ring is well known to be isomorphic to the ring of symmetric functions, and the elementary symmetric functions correspond to hooks.

Deligne categories for wreath products with a finite group or Hopf algebra were defined by Knop [8]. In [10] Mori defined wreath product Deligne categories associated to an arbitrary k-linear category $\mathcal{C}$, which is a tensor category whenever $\mathcal{C}$ is. As for the symmetric group these may be thought of as stable versions of the more classical wreath product categories in ways analogous to those listed above. See [5] for a more detailed overview of representation theory in complex rank, including discussions of the constructions mentioned here.

Motivated by a conjecture of Marin about generalizing his result to Coxeter groups in type B (which are wreath products), the goal of this paper is to prove similar results about the Grothendieck rings of these Deligne categories. By projecting from the Deligne categories to classical representation categories, we obtain systems of generators for the representation rings of wreath products with finite groups, answering the conjecture of Marin in the case when the finite group is cyclic of order 2.