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On groups generated by involutions of a semigroup

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An involution on a semigroup S (or any algebra with an underlying associative binary operation) is a function $\alpha : S \to S$ that satisfies $\alpha(xy) = \alpha(y)\alpha(x)$ and $\alpha(\alpha(x)) = x$ for all $x, y \in S$. The set $I(S)$ of all such involutions on S generates a subgroup $\mathscr{C}(S) = \langle I(S) \rangle$ of the symmetric group $\mathrm{Sym}(S)$ on the set S. We investigate the groups $\mathscr{C}(S)$ for certain classes of semigroups S, and also consider the question of which groups are isomorphic to $\mathscr{C}(S)$ for a suitable semigroup S.

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1. Introduction

Involutions are ubiquitous in many branches of mathematics, and have played a particularly significant role in algebra. There are algebras that have (external) involution operators defined on them [2,7,15,39,43,46,61,73,76,80,84,91], as well as algebras generated by (internal) involutions such as the well-known Coxeter Groups, mapping class groups, special linear groups, and non-abelian finite simple groups [13,18,40,48,57,58,72,94]. An (internal) involution in a group is an element of order 2 (i.e., a non-identity element a that satisfies $a^2 = 1$). An (external) involution on a semigroup S (or any algebra with an underlying associative binary operation) is a function $\alpha : S \to S$ satisfying $\alpha(\alpha(x)) = x$ and $\alpha(xy) = \alpha(y)\alpha(x)$, for every $x, y \in S$. Many varieties of semigroups and algebras have involutory unary operations built into their signature, including the classes of groups, inverse semigroups [61], cellular algebras [39], C^*-algebras [2,32,33,91], MI-groups [7], and regular $*$-semigroups [76]; the latter class models (for example) several diagram monoids [5,6,23–25,28,35,59,66,70].

Some well-known algebras have multiple involutions defineable on them. For example, the inverse and transpose maps $A \mapsto A^{-1}$ and $A \mapsto A^{\mathrm{T}}$ both define involutions on the general linear group $\mathrm{GL}(n, F)$, which consists of all invertible $n \times n$ matrices over a field F. The composition of these two involutions (i.e., the map $A \mapsto (A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$) is a non-inner automorphism of $\mathrm{GL}(n, F)$. Commuting involutions on a semisimple algebraic group (such as the inverse and transpose maps on the special linear group $\mathrm{SL}(n, F) \subseteq \mathrm{GL}(n, F)$) yield a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading on the associated Lie algebra [78]. Scheiblich [87] gave examples of bands (idempotent semigroups) for which two involutions give rise to non-isomorphic regular $*$-semigroups. Auinger et al. [5] studied two different involutions on the finite partition monoids (and related diagram monoids) in the context of (inherently) non-finitely based equational theories; one of these involutions leads to a regular $*$-semigroup structure and the other does not. Winker et al. [93] gave examples of semigroups with anti-automorphisms but no involution (they showed the minimal size of such a semigroup is 7, and that there are four such semigroups of minimal size, all of which are 3-nilpotent). Ciobanu et al. [11] made crucial use of a free monoid with involution in their work on word equations in free groups. Bacovský [7] investigated a class of monoids with involution that have applications in processor networks and fuzzy numbers. Gustafson et al. [40] showed that any square matrix of determinant ± 1 over any field is the product of (at most) four involutions; a square matrix is the product of two involutions if and only if it is invertible and similar to its own inverse [18,94]. A linear bound for products of involutory integer matrices was given in [51]. Everitt and Fountain studied partial mirror symmetries by investigating certain factorizable inverse monoids generated by "partial reflections" of a space [30,31]; see also [27]. Lusztig and Vogan investigated certain actions on Hecke algebras via involutions on Coxeter groups that permute the simple reflections [64,65]. Finally, we must mention the vast body of work of authors such as Dolinka, Imaoka, Jones, Petrich, Reilly, Scheiblich and Yamada