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Almost perfect restriction semigroups

Peter R. Jones

Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, WI 53201, USA

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ABSTRACT

We call a restriction semigroup *almost perfect* if it is proper and the least congruence that identifies all its projections is perfect. We show that any such semigroup is isomorphic to a 'W-product' $W(T, Y)$, where T is a monoid, Y is a semilattice and there is a homomorphism from T into the inverse semigroup TI_Y of isomorphisms between ideals of Y. Conversely, all such W-products are almost perfect. Since we also show that every restriction semigroup has an easily computed cover of this type, the combination yields a 'McAlister-type' theorem for all restriction semigroups. It is one of the theses of this work that almost perfection and perfection, the analogue of this definition for restriction monoids, are the appropriate settings for such a theorem. That these theorems do *not* reduce to a general theorem for inverse semigroups illustrates a second thesis of this work: that restriction (and, by extension, Ehresmann) semigroups have a rich theory that does not consist merely of generalizations of inverse semigroup theory. It is then with some ambivalence that we show that all the main results of this work easily generalize to encompass *all* proper restriction semigroups.
The notation $W(T, Y)$ recognizes that it is a far-reaching generalization of a long-known similarly titled construction. As a result, our work generalizes Szendrei's description of almost factorizable semigroups while at the same time

E-mail address: peter.jones@mu.edu.

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including certain classes of free restriction semigroups in its realm.

1. Introduction

The study of the structure of restriction semigroups has in large part been motivated by consideration of structure theorems for inverse semigroups. For instance, the Munn representation of inverse semigroups by isomorphisms between the principal ideals of its semilattice of idempotents generalizes naturally [6] to representations of restriction semigroups by similar mappings of its semilattice of projections, and these generalized representations are at the very heart of our work. The 'inductive groupoid' approach to inverse semigroups has been extended successfully to restriction semigroups by Lawson [16].

Somewhat complementary to the Munn representation is the McAlister theory, whereby every inverse semigroup is an idempotent-separating image of an E-unitary inverse semigroup, and the latter semigroups are described as 'P-semigroups', relative to their semilattices, greatest group images and one further structural parameter. This theory has been extended with success to restriction semigroups, with E-unitariness replaced by 'properness' and pairs of actions replacing a single one.

When moving yet further from inverse semigroups, Branco, Gomes and Gould [1,7] introduced the notion of T-properness of (one-sided) Ehresmann semigroups, with respect to a submonoid T. The main thesis of our work is that (returning to the realm of restriction semigroups) a modification of this idea yields narrower notions of properness that are the appropriate ones in which to prove a 'McAlister-type' theory. That this theory specializes in the case of proper inverse semigroups to a very narrow subclass we take to be a witness to our thesis, rather than the contrary. Our results suggest that the road taken for Ehresmann semigroups in the cited papers is indeed a natural one.

To illustrate that this is not merely a conceit, we state the main result of this paper, to illustrate its simplicity. In fact, we prove a more general theorem, applying to all proper restriction semigroups, about which more will be said below. Recall first that for restriction semigroups, monoids, considered as restriction semigroups with a single projection, play the role that groups play for inverse semigroups and that such a semigroup is *proper* if the least congruence σ that identifies all the projections (loosely, the least monoid congruence) meets each class of the generalized Green relations $\widetilde{\mathcal{L}}$ and $\widetilde{\mathcal{R}}$ trivially.

We call a restriction semigroup S *almost perfect* if it is proper and σ is perfect (meaning that the product of classes is again a class). The reason for the qualifier 'almost' is that we term a restriction *monoid* M *perfect* if, further, each σ-class has a greatest element,