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On the structure of subsets of an orderable group with some small doubling properties

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ABSTRACT

The aim of this paper is to present a complete description of the structure of subsets S of an orderable group G satisfying $|S^2| = 3|S| - 2$ and $\langle S\rangle$ is non-abelian.

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1. Introduction

Let G denote an arbitrary group (multiplicatively written). If S is a subset of G, we define its *square* S^2 by the formula

$$S^2 = \{x_1 x_2 \mid x_1, x_2 \in S\}.$$

In the abelian context, G will usually be additively written and we shall rather speak of *sumsets* and specifically of the *double* of S, namely

$$2S = \{x_1 + x_2 \mid x_1, x_2 \in S\}.$$

Here, we are concerned with the following general problem: for two real numbers $\alpha \geq 1$ and β, determine the *structure* of S if S is a finite subset of a group G satisfying an inequality on cardinalities of the form

$$|S^2| \leq \alpha|S| + \beta$$

when α is small and $|S|$ is typically large.

Problems of this kind are called *inverse problems* of *small doubling* type in additive number theory. The coefficient α (or more precisely the ratio $|S^2|/|S|$) is called the *doubling coefficient* of S. This type of problems became the most central issue in additive combinatorics. Inverse problems of small doubling type have been first investigated by G.A. Freiman very precisely in the additive group of the integers (see [4–7]) and by many other authors in general abelian groups, starting with M. Kneser [16] (see, for example, [15,17,1,23,14]). More recently, small doubling problems in non-necessarily abelian groups have been also studied, see [13,24] and [3] for recent surveys on these problems and [19] and [26] for two important books on the subject.

It is easy to prove that if S is a finite subset of $\mathbb{Z}$, then

$$2|S| - 1 \leq |2S| \leq \frac{|S|(|S|+1)}{2}.$$

Moreover $|2S| = 2|S| - 1$ if and only if S is a (finite) arithmetic progression, that is, a set of the form

$$\{a, a+q, a+2q, \ldots, a+(t-1)q\}$$

where a, q and t are three integers, $t \geq 1$, $q \geq 0$. The parameter t is called the *size* of the arithmetic progression and q its *difference* (we shall use *ratio* in the multiplicative notation). In the articles [4] and [5], G.A. Freiman proved the following more general results. The first result is referred to as the $3k-4$ theorem.

Theorem A. *Let S be a finite set of integers with at least three elements. If $|2S| \leq 3|S|-4$, then S is contained in an arithmetic progression of size $|2S| - |S| + 1 \leq 2|S| - 3$.*