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Core of ideals in integral domains ☆

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To Evan Houston on the occasion of his sixty-fifth birthday

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This paper uses objects and techniques from multiplicative ideal theory to develop explicit formulas for the core of ideals in various classes of integral domains (not necessarily Noetherian). We also investigate the existence of minimal reductions (originally established by Rees and Sally for local Noetherian rings). All results are illustrated by original examples in Noetherian and non-Noetherian settings, where we explicitly compute the core and validate some open questions recently raised in the literature.

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1. Introduction

All rings considered are integral domains (i.e., commutative with identity and without zero-divisors). Let R be a domain and I an ideal of R. An ideal J is a reduction of I if $J \subseteq I$ and $JI^n = I^{n+1}$ for some positive integer n. This notion was introduced by D.G. Northcott and D. Rees [44] and has recently played a crucial role in the study of Rees algebras of ideals. The notion of core of an ideal, denoted core(I), and defined as the intersection of all reductions of I, was introduced by Judith Sally in the late 1980s and was alluded to in Rees and Sally's paper [49]. The core of an ideal naturally appears also in the context of Briancon–Skoda's Theorem; a simple version of which states that if R is a d-dimensional regular ring and I is any ideal of R, then the integral closure of I^d is contained in core(I).

In 1995, Huneke and Swanson [31] determined the core of integrally closed ideals in two-dimensional regular local rings and established a correlation to Lipman's adjoint ideal. Recently, in a series of papers [12,13,47], Corso, Polini and Ulrich gave explicit descriptions for the core of certain ideals in Cohen–Macaulay local rings, extending the results of [31]. In 1997, Mohan [43] investigated the core of a module over a two-dimensional regular local ring and was inspired by the original work of Huneke and Swanson. In 2003, Corso, Polini and Ulrich [14] determined the core of projective dimension one modules and recovered, in particular, the result by Mohan. In 2003, Hyry and K.E. Smith [34] generalized the results in [31] to arbitrary dimensions and more general rings. In 2005, Huneke and Trung [33] answered several open questions raised by Corso, Polini and Ulrich. In 2007, Polini, Ulrich, and Vitulli [48] gave some remarkable results on the computation of the core of zero-dimensional monomial ideals. In 2008, Fouli, Polini and Ulrich [20] studied the core in arbitrary characteristic and, in 2010, the same authors [21] investigated the annihilators of graded components of the canonical module and the core of standard graded algebras. In this latter paper, for example, the authors characterized Cayley–Bacharach sets of points in terms of the structure of the core of the maximal ideal of their homogeneous coordinate ring. Finally, in 2011, B. Smith [51] established a formula for the core of certain strongly stable ideals that satisfy some local properties and used a result of Polini and Ulrich which showed that the core of such an ideal I is the largest monomial ideal contained in K, for any general minimal reduction K of I.

As the intersection of an a priori infinite number of ideals, the core seems extremely difficult to determine and there are few computed examples in the literature. Little is known about the structure and the properties of core(I) and most of the works on this topic were done in the Noetherian case; precisely, in Cohen–Macaulay rings, where minimal reductions of an ideal exist and have pleasing property of carrying most of the information about the origin ideal.

A domain R is said to have the trace property if for each nonzero ideal I of R, either I is invertible in R or $I(R : I)$ is a prime ideal of R [17,18,41]. Valuation domains [2, Theorem 2.8] and pseudo-valuation domains [29, Example 2.12] have the trace property.